ECE 520.651 Random Signal Analysis

Homework # 12

Due 12:00 Noon on Monday, November 29, 2004.

1. Fill in the details of the following topics covered in class.

   (a) On the bottom of page 227 in the Poor book, clearly show how

   \[ E \left[ (X_l - E[X_l]) Y_l \right] = \sum_{n=a}^{b} h_{l,n} E \left[ (Y_n - E[Y_n]) Y_l \right], \quad a \leq l \leq b, \]

   implies

   \[ \text{Cov} \left( X_l, Y_l \right) = \sum_{n=a}^{b} h_{l,n} \text{Cov} (Y_n, Y_l), \quad a \leq l \leq b. \]

   (b) In finding the LMMSE estimate of \( X \) based on \( Y_a^b = \{Y_a, \ldots, Y_b\} \), make a precise argument why we can assume without loss of generality that \( X \) and \( Y_a^b \) are zero mean random variables. Specifically, if any of \( E[X] = \mu_X \) or \( E[Y_a^b] = \mu_a^b \) are nonzero, then show how to

   i. construct random variables \( Z \) and \( W_a^b \) which are zero mean,
   ii. construct the LMMSE estimate \( \hat{Z} \) of \( Z \) based on \( W_a^b \), and
   iii. (re)construct the LMMSE estimate \( \hat{X} \) of \( X \) from \( \hat{Z} \).

   (c) Do the arguments of 1b carry through for the MMSE estimate of \( X \)?

2. Alternate proof of Theorem 9.1-4(a) from Stark and Woods: Let \( \hat{X}_1 \) be the LMMSE estimate of \( X_1 \) based on \( Y = [Y_a \ldots Y_b]^T \), and \( \hat{X}_2 \) the LMMSE estimate of \( X_2 \) based on \( Y \). Use the orthogonality principle to show that the LMMSE estimate \( \hat{Z} \) of

   \[ Z = X_1 + X_2 \]

   based on \( Y \) satisfies:

   \[ \hat{Z} = \hat{X}_1 + \hat{X}_2 \quad \text{almost surely.} \]

   Hint: Compute the difference

   \[ E \left[ (Z - \hat{Z})^2 \right] - E \left[ (Z - (\hat{X}_1 + \hat{X}_2))^2 \right] \]

   and use the fact that linear combinations of any two (or all three) of \( \hat{X}_1, \hat{X}_2 \) and \( \hat{Z} \) are in \( \mathcal{H}_a^b \), the family of affine functions of \( Y \).
3. Alternate proof of Theorem 9.1-4(b) from Stark and Woods: Let \( \hat{X}_1 \) be the LMMSE estimate of \( X \) based on \( Y_1 = [Y_a \ldots Y_b]^T \), and \( \hat{X}_2 \) the LMMSE estimate of \( X \) based on \( Y_2 = [Y_c \ldots Y_d]^T \). Use the orthogonality principle to show that if \( Y_1 \) and \( Y_2 \) are orthogonal random vectors, then the LMMSE estimate \( \hat{X} \) of \( X \) based on

\[
Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}
\]

satisfies

\[
\hat{X} = \hat{X}_1 + \hat{X}_2 \quad \text{almost surely.}
\]

Hint: Assume all random variables are zero mean and, following (V.C.19) on p228 in the Poor book, write the LMMSE estimate of \( X \) based on \( Y \) as

\[
\hat{X} = \mathbf{h}^T Y = \begin{bmatrix} \mathbf{h}_1^T \\ \mathbf{h}_2^T \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \mathbf{h}_1^T Y_1 + \mathbf{h}_2^T Y_2,
\]

where \( \mathbf{h} = \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{bmatrix}, \mathbf{h}_1^T Y_1 \in \mathcal{H}_a^b \) and \( \mathbf{h}_2^T Y_2 \in \mathcal{H}_c^d \). If \( Y_1 \perp Y_2 \), does it follow that the (scalar) random variables \( \mathbf{h}_1^T Y_1 \) and \( \mathbf{h}_2^T Y_2 \) are orthogonal for every \( \mathbf{h} \)?

4. Consider the problem of estimating a random variable \( X \), with \( E[X^2] < \infty \), based on the random sequence \( \{Y_i\}_{i=0}^\infty \). For any fixed \( n \), the MMSE estimate of \( X \) based on \( Y_0^n \) is a well defined random variable:

\[
Z_n = E[X | Y_0^n].
\]

Show that the sequence \( \{Z_n\}_{n=0}^\infty \) is a Martingale.

5. Use the Martingale Convergence Theorem (Theorem 6.8-4 in Stark and Woods) to prove that the sequence \( Z_n \) defined in Problem 4 converges almost surely. Hint: Try to bound \( E[Z_n^2] \) by \( E[X^2] \).

Remark: This (almost sure) limit is defined as the MMSE estimate of \( X \) based on the entire sequence \( \{Y_i\}_{i=0}^\infty \).