ECE 520.651 Random Signal Analysis

Homework # 11

1. (IV.F.16 from the Poor book) Let $\theta > 0$ be an unknown parameter, and $Y_1, \ldots, Y_n$ be a sequence of $\mathbb{R}$-valued i.i.d. random variables with common density

$$p_\theta(y) = \begin{cases} \frac{e^{-\frac{y^M}{\theta^M}}}{(2\theta)^M M!} & \text{if } y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

where $M$ is a known positive integer.

(a) Find the ML estimate of $\theta$ based on $Y_1, \ldots, Y_n$.
(b) Compute the bias and variance of the estimate from 1a.
(c) Compute the Cramér-Rao bound on the variance of unbiased estimates of $\theta$.
(d) Is the ML estimate of part 1a consistent? Is it efficient?

2. (IV.F.17 from the Poor book) Let $Y_1$ and $Y_2$ be jointly Gaussian, each with zero mean, unit variance and an unknown correlation coefficient $\rho = E[Y_1 Y_2]$.

(a) Find the equation for the ML estimate of $\rho$ based on $(Y_1, Y_2)$.
(b) Compute the Cramér-Rao bound for the variance of unbiased estimates of $\rho$.

3. (IV.F.18 from the Poor book) Suppose that we observe

$$Y_k = N_k + \theta s_k, \quad k = 1, \ldots, n,$$

where $\mathbf{N} = [N_1 \ldots N_n]^T$ is a zero-mean Gaussian random vector with $n \times n$ covariance matrix $\mathbf{\Sigma}$, $s_1, \ldots, s_n$ is a known signal sequence, and $\theta \in \mathbb{R}$ is a nonrandom parameter.

(a) Find the ML estimate of $\theta$ based on $Y_1, \ldots, Y_n$.
(b) Compute the bias and variance of the estimate of 3a.
(c) Compute the Cramér-Rao bound for unbiased estimates of $\theta$ and compare with your result from 3b.
(d) What can be said about the consistency of $\hat{\theta}_{ML}$ as $n \to \infty$? Specifically, does consistency follow if

$$
\frac{1}{n} \sum_{k=1}^{n} s_k^2 > a \quad \forall \ n \quad \text{and} \\
\lambda_{\min}(\Sigma^{-1}) > b \quad \forall \ n,
$$

for some positive constants $a$ and $b$, where $\lambda_{\min}(A)$ denotes the smallest eigenvalue of a matrix $A$?

4. (IV.F.19 from the Poor book) Suppose $\theta > 0$ is a nonrandom parameter and we observe a sequence $Y_1, \ldots, Y_n$ given by

$$
Y_k = \sqrt{\theta} N_k \quad k = 1, \ldots, n,
$$

where $N = [N_1 \ldots N_n]^T$ is a zero-mean Gaussian random vector with a positive definite covariance matrix $\Sigma$.

(a) Find the ML estimate of $\theta$ based on $Y_1, \ldots, Y_n$.

(b) Show that the estimate of 4a is unbiased.

(c) Compute the Cramér-Rao bound for unbiased estimates of $\theta$.

(d) Compute the variance of the estimate of 4a and compare with the CRLB from 4c.

5. (IV.F.20 from the Poor book) Consider the observation model

$$
Y_k = \sqrt{\theta} s_k R_k + N_k, \quad k = 1, \ldots, n,
$$

where $s_1 \ldots, s_n$ is a known signal sequence, $N_1, \ldots, N_n, R_1, \ldots, R_n$ are i.i.d. $\mathcal{N}(0, 1)$ random variables, and $\theta > 0$ is an unknown nonrandom parameter.

(a) Find the likelihood equation for estimating $\theta$ from $Y_1, \ldots, Y_n$.

(b) Find the Cramér-Rao bound on the variance of unbiased estimates of $\theta$.

(c) Find the MLE explicitly for the special case when $s_1, \ldots, s_n$ is a sequence of $\pm 1$’s.

(d) Compute the bias and variance of your estimate from 5c and compare with the CRLB from 5b.