Solution to Homework 5-Random Signal Analysis

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1 5.10

We know that a linear transformation of a gaussian random variable is a gaussian random variable.
Also we know that the eigen vectors of a positive-symmetric matrix are orthonormal.

\[ \mathbf{E}[\mathbf{Y}^\mathbf{T}] = \mathbf{CE}[\mathbf{X}^\mathbf{T}]\mathbf{C}^\mathbf{\top} = \mathbf{CK}_{\mathbf{XX}}\mathbf{C}^\mathbf{\top} \]  

which means

We can easily find the eigen values of \( \mathbf{K}_{\mathbf{XX}} \) which are 2 and 5.
The corresponding eigen vectors are \([1, \sqrt{2}]^\mathbf{\top}\) and \([-\sqrt{2}, 1]^\mathbf{\top}\).

From matrix diagonalisation, we have

\[
\begin{pmatrix}
-\frac{\sqrt{2}}{\sqrt{3}} & 1 \\
\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}}
\end{pmatrix}
\begin{pmatrix}
3 & \sqrt{2} \\
\sqrt{2} & 4
\end{pmatrix}
\begin{pmatrix}
-\frac{\sqrt{2}}{\sqrt{3}} & 1 \\
\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}}
\end{pmatrix}
= \begin{pmatrix}
2 & 0 \\
0 & 5
\end{pmatrix}
\]

So if the orthonormal eigen vectors \( e_i \) are divided by \( \sqrt{\lambda_i} \), we get Identity on the RHS. This means that

\[
\begin{pmatrix}
-\frac{1}{\sqrt{3}} & 1 \\
\frac{1}{\sqrt{15}} & \frac{\sqrt{2}}{\sqrt{15}}
\end{pmatrix}
\begin{pmatrix}
3 & \sqrt{2} \\
\sqrt{2} & 4
\end{pmatrix}
\begin{pmatrix}
-\frac{1}{\sqrt{3}} & 1 \\
\frac{1}{\sqrt{6}} & \frac{\sqrt{7}}{\sqrt{15}}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

Comparing with equation (1), we have the required matrix as

\[ \mathbf{C} = \begin{pmatrix}
-\frac{\sqrt{3}}{\sqrt{15}} & \frac{\sqrt{2}}{\sqrt{15}} \\
\frac{1}{\sqrt{15}} & \frac{\sqrt{2}}{\sqrt{15}}
\end{pmatrix} \]

Clearly \( \mathbf{C} \) is not a unitary transformation as \( \text{Det}(\mathbf{C}) \neq 1 \).

1.1 Part b

In order for \( \mathbf{AA}' \) to be real symmetric we need the elements (1,2) and (2,1) to be same. This means that \( ab' + bc' \) and \( a'b + b'c \) must be same. But \( ab' = cb' \) and \( bc' = ba' \) since \( a = c \) and \( a' = c' \). This means that \( ab' + bc' = cb' + ba' = a'b + b'c \). Hence the elements (1,2) and (2,1) are same. Hence \( \mathbf{AA}' \) is a real symmetric matrix.

1
Since A and A’ are symmetric matrices, we have $A = A^T$ and $A' = (A')^T$

In order for $AA'$ to be symmetric, we need

$$AA' = (AA')^T = (A')^T A^T = A'A$$

ie we need $AA' = A'A$ and is sufficient for $AA'$ to be symmetric if A and A’ are symmetric.

## 2 Problem 5.12

### 2.1 Part a

Note that $WW^T$ can be written as a multiplication of 1 row vector and 1 column vector (each element with dimension n), which means

$$WW^T = [X_1 X_2 X_3 \ldots X_m][X_1^T X_2^T X_3^T \ldots X_m]^T$$

So we get

$$WW^T = \sum_{i=1}^{m} X_i X_i^T$$

Hence

$$S = \frac{1}{m} WW^T$$

### 2.2 Part b

We know that

$$\text{Rank}(S) \leq \min(\text{Rank}(W)\text{Rank}(W^T)) = m$$

Hence the maximum rank is $m$

### 2.3 Part c

The size of $S'$ is $mxm$.

We have that

$$\frac{1}{m} W^T W \Phi = \Phi \Lambda$$

Premultiplying by W, we get

$$\frac{1}{m} WW^T (W \Phi) = W \Phi \Lambda$$

Since the maximum rank of $S$ is $m$, we have at most $m$ non-zero eigen values for $S$, and all these eigen values are contained in $\text{diag}(\Lambda)$ and the corresponding $m$ eigen vectors are each of the columns of $W \Phi$.

### 2.4 Part d

In order to compute the eigen-vectors of $S$, we need to get the eigen values first. For that we need to solve a n-degree characteristic equation. So if we know that the rank is $m$, it is enough to solve a m-degree equation. This m-degree equation is the characteristic function of $S'$. In short if we know that the matrix is reduced in dimensions, why should we unnecessarily go to a higher dimension and make the calculations. If we can represent some points in 2-d, there is no point adding an extra coordinate and representing them in 3-d.
3 Problem 5.20

Since we have
\[
\Phi_x(\omega_1, \omega_2, \omega_3, \omega_4) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_x(x_1, x_2, x_3, x_4) e^{j(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4)}
\]
we obtain by partial differentiating 4 times
\[
E[X_1 X_2 X_3 X_4] = \frac{1}{j^4} \frac{\partial^4 \Phi(\omega)}{\partial \omega_1 \partial \omega_2 \partial \omega_3 \partial \omega_4} = \frac{\partial^4 \Phi(\omega)}{\partial \omega_1 \partial \omega_2 \partial \omega_3 \partial \omega_4}
\]
The characteristic function of a zero-mean-gaussian is
\[
exp\left(-\frac{\omega^T K \omega}{2}\right)
\]
which is written as
\[
\Phi(\omega) = exp\left(-\frac{1}{2} \left(\sum_{i=1}^{4} \sum_{j=1}^{4} K_{ij} \omega_i \omega_j\right)\right)
\]
We shortly write it as
\[
\Phi(\omega) = exp\left(-\frac{1}{2} \left(\sum_{i,j} K_{ij} \omega_i \omega_j\right)\right)
\]
Let \(I(k = l)\) be the indicator function.It is also called the Kronecker’s delta function which is denoted as \(\delta_{kl}\). \(\delta_{kl} = 1\) if \(k = l\) otherwise it is 0.

Note the following
\[
\frac{\partial}{\partial \omega_n} \Phi(\omega) = \Phi(\omega) \left(-\frac{1}{2} \left(\sum_{i,j} K_{ij} \delta_{in} \omega_j\right) - \frac{1}{2} \left(\sum_{i,j} K_{ij} \delta_{jn} \omega_i\right)\right)
\]
\[
= -\Phi(\omega) \left(\sum_{i,j} K_{ij} \delta_{in} \omega_j\right)
\]
Denote \(\sum_{i,j} K_{ij} \delta_{in} \omega_j\) as \(\zeta_n(\omega)\)

Now
\[
\frac{\partial}{\partial \omega_m} \zeta_n(\omega) = \left(\sum_{i,j} K_{ij} \delta_{in} \delta_{jm}\right) = K_{nm}
\]
and
\[
\frac{\partial}{\partial \omega_n} \Phi(\omega) = -\Phi(\omega) \zeta_n(\omega)
\]
So we have
\[
\frac{\partial}{\partial \omega_1 \partial \omega_2 \partial \omega_3 \partial \omega_4} [\Phi(\omega)] = \\
\frac{\partial}{\partial \omega_1 \partial \omega_2 \partial \omega_3} (-\Phi(\omega)\zeta_4(\omega)) = \\
\frac{\partial}{\partial \omega_1 \partial \omega_2} (\Phi(\omega)\zeta_3(\omega)\zeta_4(\omega) - \Phi(\omega)K_{34})
\]
(2)

Now note that
\[
\frac{\partial}{\partial \omega_1 \partial \omega_2} (\Phi(\omega)\zeta_3(\omega)\zeta_4(\omega))
\]
will have 6 terms. Also note that \( \zeta_n([0,0,0,0]) = 0 \). Since finally we have to put \( \omega = [0,0,0,0] \), out of the 6 terms only 2 terms will survive and that will be the term that has got rid of both \( \zeta_3(\omega) \) and \( \zeta_4(\omega) \) which can happen only if both are differentiated and the only 2 terms among the 6 satisfying this are

\[
\Phi(\omega)\frac{\partial \zeta_3(\omega)}{\partial \omega_1} \frac{\partial \zeta_4(\omega)}{\partial \omega_2} = K_{13}K_{24}\Phi(\omega)
\]

and

\[
\Phi(\omega)\frac{\partial \zeta_3(\omega)}{\partial \omega_2} \frac{\partial \zeta_4(\omega)}{\partial \omega_1} = K_{23}K_{14}\Phi(\omega)
\]

So the value of
\[
\frac{\partial}{\partial \omega_1 \partial \omega_2} (\Phi(\omega)\zeta_3(\omega)\zeta_4(\omega))
\]
at \( \omega_1 = \omega_2 = \omega_3 = \omega_4 = 0 \) is
\( K_{13}K_{24} + K_{23}K_{14} \)

Also we have
\[
\frac{\partial}{\partial \omega_1 \partial \omega_2} (\Phi(\omega)) = \\
\frac{\partial}{\partial \omega_1} (-\Phi(\omega)\zeta_2(\omega)) = \\
\Phi(\omega)\zeta_1(\omega)\zeta_2(\omega) - \Phi(\omega)K_{12}
\]

and the value of this at \( \omega_1 = \omega_2 = \omega_3 = \omega_4 = 0 \) is simply \( -K_{12} \)

So we have the value of eqn(2) which is
\[
\frac{\partial}{\partial \omega_1 \partial \omega_2} (\Phi(\omega)\zeta_3(\omega)\zeta_4(\omega) - \Phi(\omega)K_{34})
\]
as
\[
K_{13}K_{24} + K_{23}K_{14} - K_{34}(-K_{12}) = K_{13}K_{24} + K_{23}K_{14} + K_{34}K_{12}
\]
which is what we want to show.
4 Problem 5.23

We have \( Y = BX \)
For every change \( dx \), we have a change in the \( Y \) domain which is \( Bdx \). The ratio of the volumes of changes is determined by the magnitude of the determinant which is

\[
det \left[ \frac{\partial (Bx)}{\partial x} \right] = det(B)
\]

and this the jacobian of the transformation So we have

\[
f_Y(Y) = |det(B)|f_X(By)
\]

which is same as

\[
f_Y(y_1, y_2, \ldots, y_n) = |det(B)|f_X(x_1^*, x_2^* \ldots x_n^*)
\]

5 Problem 5

5.1 Part a

For a full rank matrix \( K \), we would have \( k = n \) which means \( A \) will be a square matrix.

Using characteristic functions, it is not hard to show that any linear transformation of a gaussian random vector will be a gaussian (refer next part).

So We need to get \( A \) such that the mean of the random variable \( AY + b \) to be \( \mu \) which means \( b = \mu \).

Also we need

\[
E \left[ AYY^T A^T \right] = AKYY^T A^T = K
\]

which means

\[
AA^T = K
\]

For a square matrix (part A), \( A \) can simply be

\[
A = K^{\frac{1}{2}}
\]

5.2 Part b

We need to solve

\[
AA^T = K
\]

If \( K \) has a rank \( k < n \), we know that it will have \( m \) non-zero eigen values, and hence \( m \) eigen vectors corresponding to these eigen values. Let \( e_1, e_2, e_3, \ldots, e_k \) denote the eigen vectors and let \( \lambda_1, \lambda_2, \ldots, \lambda_k \) be the corresponding eigen values.

So we can write

\[
K = [e_1, e_2, e_3, \ldots, e_k]
\]

\[
\left( \begin{array}{ccc}
\lambda_1 & \cdots & 0 \\
0 & \lambda_2 & 0 \\
\cdots & \cdots & \cdots \\
0 & \cdots & \lambda_k
\end{array} \right)
\left( \begin{array}{c}
e_1^T \\
e_2^T \\
\vdots \\
e_k^T
\end{array} \right)
\]

where \( e_i \) are the eigen vectors (ie nX1 arrays)
So the corresponding $A$ matrix would be

$$A = \begin{bmatrix} e_1, e_2, e_3, \ldots, e_k \end{bmatrix} \begin{pmatrix} \sqrt{\lambda_1} & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\lambda_k} \end{pmatrix}$$

which will be a $n \times k$ matrix. Thus we have constructed $A$ such that

$$AA^T = K$$

6 Problem 6

We have for a gaussian random variable $X$

$$E\left[e^{\omega^T x}\right] = exp \left[j\omega^T \mu - \frac{1}{2} \omega^T K \omega \right]$$

The characteristic function for $Y = AX$ is given by

$$E\left[e^{\omega' T y}\right] = E\left[e^{\omega' T Ax}\right] = E\left[e^{(A^T \omega') T x}\right]$$

Note that $\omega'$ is a $m \times 1$ vector while $\omega$ is a $n \times 1$ vector.

which means that the characteristic function of $Y$ is obtained by replacing $\omega'$ by $A^T \omega$ in the characteristic function of $X$.

Noting that $(A^T \omega')^T = \omega'^T A$

we obtain

$$\Phi_y(\omega) = E\left[e^{\omega' T y}\right] = exp \left[j\omega'^T A \mu - \frac{1}{2} \omega'^T AKA^T \omega' \right]$$

Comparing with the standard form for the characteristic function of a gaussian, we obtain

$$Q = AKA^T$$

and

$$\beta = A \mu$$