1 3.28-Stark and Woods

The expression of the inverse Jacobian which is

\[
\begin{vmatrix}
\frac{\partial Z}{\partial X} & \frac{\partial Z}{\partial W} \\
\frac{\partial Y}{\partial X} & \frac{\partial Y}{\partial W}
\end{vmatrix}
\]

which (on direct computation) simplifies to \(2Y\) which is written in terms of \(Z\) and \(W\) as

\[|J| = 2\sqrt{Z - W^2}\]

Notice the fact the the pair \(Z,W\) can be generated using 2 different pairs \((X, Y)\) and \((X, -Y)\). So we need to set a multiplication factor of 2.

So the joint pdf is simply

\[
f_{zw}(z, w) = \frac{1}{2\pi\sigma^2} e^{-z/2\sigma^2} \frac{2}{|J|}
\]

which becomes

\[
f_{zw}(z, w) = \frac{1}{2\pi\sigma^2\sqrt{z - w^2}} e^{-z/2\sigma^2}
\]

We also need to define one more thing,—the ranges of \(z\) and \(w\).

Clearly \(w\) can vary from \(-\sqrt{z}\) to \(+\sqrt{z}\). This also comes from the fact that \(x^2 + y^2 \geq x^2\).

\(z\) varies from 0 to \(\infty\). It does not take negative values because it is \(x^2 + y^2\).

To get \(f_z(z)\) we need to marginalise out the pdf wrt \(w\)

\[
\int_{w=-\sqrt{z}}^{w=\sqrt{z}} \frac{1}{2\pi\sigma^2\sqrt{z - w^2}} e^{-z/2\sigma^2} = \frac{1}{2\sigma^2} e^{-z/2\sigma^2}
\]

2 Problem 4.30

We are required to find the cdf

\[
f_X(x) = \frac{\alpha}{\pi(\alpha^2 + x^2)}
\]

First let us revise the properties of fourier transform.

We know that if

\[
\int_{-\infty}^{+\infty} f(x)e^{-j\omega x} dx = F(j\omega)
\]

then

\[
\int_{-\infty}^{+\infty} F(j\omega)e^{j\omega x} dw = 2\pi f(x)
\]

Setting \(F(j\omega) = exp(-\alpha\omega)u(\omega)\), we get

\[
2\pi f(x) = \frac{1}{jx - \alpha}
\]
that is

\[
f(x) = \frac{1}{2\pi} \frac{1}{jx - \alpha}
\]

In other words

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{jx - \alpha} e^{-j\omega x} dx = \exp(-\alpha \omega) u(\omega)
\] (1)

Replacing \( \omega \) by \(-\omega\) we get

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{jx + \alpha} e^{j\omega x} dx = \exp(\alpha \omega) u(-\omega)
\]

Flipping the limits of integration, we see that this integral is same as

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{-1}{jx + \alpha} e^{-j\omega x} dx = \exp(\alpha \omega) u(-\omega)
\] (2)

Adding eq(1) and eq(2), we get

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\alpha}{x^2 + \alpha^2} dx = \exp(-\alpha |\omega|)
\]

hence the characteristic function of a Cauchy is an exponential on both sides, i.e., it is \( \exp(-\alpha |\omega|) \)

3 Part 4.31

Since \( X_1, X_2 \ldots X_N \) are iid, the characteristic function of \( Y \) can be written as

\[
\phi_y(\omega) = \prod_{i=1}^{N} \phi_x(\omega/N)
\]

We know from our previous derivation that

\[
\phi_x(\omega/N) = \exp(-\alpha |\omega|/N) \exp(j\omega \mu)
\]

Hence

\[
\prod_{i=1}^{N} \phi_x(\omega/N) = \exp(-\alpha |\omega|/N * N) * \exp(j\omega \mu N/n) = \exp(-\alpha |\omega|) \exp(j\omega \mu)
\]

Hence the cf of \( Y \) is also

\[
\exp(-\alpha |\omega|) \exp(j\omega \mu)
\]

Since the characteristic function has a unique inverse, we would have

\[
f_Y(x) = \frac{1}{\pi [1 + (x - \mu)^2]}
\]
4 Problem 4.45 Stark and Woods

Let us first compute the characteristic function of the random variable

\[ Y = X^2 \]

The moment generating function can be written as

\[ E(e^{tX^2}) = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2\sigma^2} e^{tx^2} dx = \sqrt{\frac{1}{1-2t\sigma^2}} \]

Hence the moment generating function for the random variable obtained by

\[ Y = \sum_{i=1}^{n} X_i^2 \]

is simply

\[ \frac{1}{(1-2t\sigma^2)^{n/2}} \]

and the characteristic function is simply obtained by relacing \( t \) by \(-j\omega\) Hence it is

\[ \frac{1}{(1 + 2j\omega\sigma^2)^{n/2}} \]