1 Problem 1

1.1 Non-negativity

Since $\lambda, 1 - \lambda, P_1(A), P_2(A)$ are non-negative quantities $P_\lambda(A) \geq 0$

1.2 Normalization

$$\sum_{A \in \mathcal{F}} P_\lambda(A) = \lambda \sum_{A \in \mathcal{F}} P_1(A) + (1 - \lambda) \sum_{A \in \mathcal{F}} P_2(A) = \lambda + 1 - \lambda = 1$$

1.3 Countable Additivity

$$P_\lambda(\bigcup_{i=1}^n A_i) = \lambda P_1(\bigcup_{i=1}^n A_i) + (1 - \lambda)P_2(\bigcup_{i=1}^n A_i)$$

If $A_1, A_2 \ldots A_n$ are pairwise disjoint events, we have

$$P_1(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n P_1(A_i)$$

and

$$P_2(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n P_2(A_i)$$

So

$$P_\lambda(\bigcup_{i=1}^n A_i) = \lambda \sum_{i=1}^n P_1(A_i) + (1 - \lambda) \sum_{i=1}^n P_2(A_i) = \sum_{i=1}^n (\lambda P_1(A_i) + (1 - \lambda)P_2(A_i)) = \sum_{i=1}^n P_\lambda A_i$$

2 Problem 2

Let $\omega$ be the random variable that has probability distribution $\mathcal{F}$ $P(X \in (a, b]) = b - a$

If we define $X_1$ according to the following equation

$$X_1 = \begin{cases} -1 & \text{if } \omega < \frac{1}{4} \\ 0 & \text{if } \frac{1}{4} < \omega \leq \frac{3}{4} \\ 1 & \text{if } \frac{3}{4} < \omega \leq 1 \end{cases}$$

we have $P(X_1 = -1) = \frac{1}{4}, P(X_1 = 0) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$ and $P(X_1 = 1) = \frac{1}{4}$

3 Problem 3

The simplest way of doing it is to shift and scale $\omega$ such that it takes values between -2 and +2 In other words

$$X_2 = 4\omega - 2$$
4 Problem 4

In order to be independent \( f(X_2/X_1) = f(X_2) \)

But according to our previous definitions, we know that if \( X_1 \) takes -1, \( X_2 \) has to be between -1 and +1. ie the probability of it taking values in the interval \([-2, -1) \cup (1, 2]\) is 0. Hence \( X_2 \) is not independent of \( X_1 \).

To make it independent, we need to set \( f(X_2/X_1) = f(X_2) \)

In other words \( X_2 \) will take values between -2 and +2 uniformly irrespective of the value of \( X_1 \). The obvious way to do this is to expand each interval uniformly to span the entire \([-2, 2]\) space

\[
X_2 = \begin{cases} 
16\omega - 2 & \text{if } \omega < \frac{1}{4} \\
8\omega - 4 & \text{if } \frac{1}{4} < \omega \leq \frac{3}{4} \\
16\omega - 14 & \text{if } \frac{3}{4} < \omega \leq 1
\end{cases}
\]

Now \( X_2 \) and \( X_1 \) are independent

5 Problem 5-2.24 in Stark and Woods

5.1 Part a

The sample space consists of the events G, R and Y. The 5 events would be G, R, Y \( G \cup R \cup Y \)

5.2 Part b

The probability distribution of X would be \( P(X=-1)=1/4 \)
\( P(X=0)=1/2 \)
\( P(X=\pi)=1/4 \)

\( P(X \leq 3)=P(G)+P(R)=3/4 \)