Submit your solutions to the following problems:


   (d) In your answer to 6.41(c), point out where the conclusion fails if $x$ is not a point of continuity of $F_X$.

2. Show that if \( \sum_{n=1}^{\infty} E[|X_n - X|^r] < \infty \) for some $r > 0$, then $X_n \to X$ almost surely.

   Loosely said, if $X_n$ converges in the $r$-th mean sufficiently fast, then it converges a.s.

3. The second Borel-Cantelli lemma: Show that if $A_n$ is a sequence of independent events for which

   \[
   \sum_{n=1}^{\infty} \mathbb{P}(A_n) = +\infty, \quad \text{then} \quad \mathbb{P}\left( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) = 1.
   \]

   Construct a counterexample to show that the lemma does not hold if the $A_n$’s are not independent.

4. Strong Law of Large Numbers: The proof on p78 in Papamarcou states that, “Using the same arguments with $-X_i$ replacing $X_i$, ... we infer that ...”

   \[
   \mathbb{P}\left\{ \frac{X_1 + \ldots + X_n}{n} \leq -\alpha \right\} \leq \rho^n.
   \]

   Use these “same arguments” to construct a clear proof of this inequality.

5. Strong Law of Large Numbers: We assumed in the proof in Papamarcou p77 that $\mu = 0$. Re-derive the proof without this assumption. Specifically, begin with the probability calculation at the top of p78, i.e.

   \[
   \mathbb{P}\left\{ \frac{X_1 + \ldots + X_n}{n} - \mu \geq \alpha \right\}, \quad \text{which can be shown to be} \quad = (\text{something})^n,
   \]

   and work your way through to the assertion on that

   \[
   \frac{d}{ds}(\text{something})\bigg|_{s=0} = -\alpha < 0.
   \]
Will this calculation suffice to then continue as before and claim that:

$$\Pr \left\{ \left| \frac{X_1 + \ldots + X_n}{n} - \mu \right| \geq \alpha \right\} \leq 2\rho^n ?$$

After finishing the homework, discuss with your study-mates the Remarks on pages 73, 76 and 80 in Papamarcou. They contain interesting mathematical claims whose proofs would make for nice exam problems!