Submit your solutions to the following problems:

1. Consider two urns A and B containing a total of N balls. A sequence of experiments is performed at times \( n = 1, 2, \ldots \), in which a ball is drawn, a destination urn is selected, and the drawn ball is placed in the selected urn.

Let \( X[n] \) denote the number of balls in urn A after the \( n \)-th experiment, with \( X[0] \) being understood as the number of balls in urn A before the experiments begin.

For a fixed \( p \in (0, 1) \) and \( q = 1 - p \), consider the following scenarios. At time \( n \),

(i) a ball is drawn uniformly at random from the totality of N balls;
   the destination urn is selected: A with probability \( p \) or B with \( q \).

(ii) the destination urn is selected: A with probability \( \frac{X[n-1]}{N} \) or B with \( \frac{N-X[n-1]}{N} \);
   a ball is drawn from urn A with probability \( p \) or urn B with \( q \).

(iii) the destination urn is selected: A with probability \( \frac{X[n-1]}{N} \) or B with \( \frac{N-X[n-1]}{N} \);
   a ball is drawn from urn A with probability \( \frac{X[n-1]}{N} \) or urn B with \( \frac{N-X[n-1]}{N} \).

Assume that the drawing of the ball and the selection of the destination urn are independent, and that the drawing, section and placement are instantaneous, i.e. no time elapses while performing them.

Discuss whether \( \{X[n]\} \) is a Markov chain in each of the three scenarios. For each scenario where it is a Markov chain,

(a) construct the one-step transition probability matrix;
(b) draw the state transition diagram;
(c) determine if the Markov chain is irreducible;
(d) determine if the Markov chain is aperiodic.

2. In each scenario where \( \{X[n]\} \) is a Markov chain, set \( N = 3 \) and solve \( \pi = \pi P \), subject to \( \pi \geq 0 \) and \( \|\pi\|_1 = 1 \). Explain if there is no solution or the solution is not unique.

3. Let the active life of a component in a machine be a discrete random variable \( T \), with probability mass function \( \mathbb{P}(T = k) = p_k, \ k = 1, 2, \ldots \).
Let \( X[n] \) denote the age of the component in service at time \( n = 0, 1, 2, \ldots \), and assume that one starts with a new component, so that \( X[0] = 0 \).

Assume further that the component is instantly replaced upon failure by a new component. For instance, if the first component has life \( k \), then

\[
\begin{align*}
X[0] &= 0, \\
X[1] &= 1, \\
\vdots & \\
X[k-1] &= k - 1, \text{ followed by component failure at time } k, \text{ and} \\
X[k] &= 0, \text{ because the component has just been replaced}, \\
\vdots & \\
\end{align*}
\]

and so on. Assume that the replacement has an independent and identically distributed active life.

Argue that \( \{X[n]\} \) is a Markov chain, determine its one-step transition probability matrix and draw its state transition diagram. Characterize its states, and determine whether the chain admits a steady-state distribution.

Hint: it may be useful to compute \( \lambda_k = \mathbb{P}(T \geq k + 1 \mid T \geq k) \) to simplify notation.

4. A maintenance schedule is implemented so that the component is replaced either upon failure or when its age reaches \( N \), whichever occurs first. In other words, the effective active life of a component is now \( T^* = \min\{T, N\} \).

Argue that \( \{X[n]\} \) is still a Markov chain, determine its one-step transition probability matrix and draw its state transition diagram. Characterize its states, and determine whether the chain admits a steady-state distribution.

Compute \( E[T^*] \).

5. Describe a research problem you expect to encounter in your PhD program—in your own research or in some paper you read—in which a Markov chain is a reasonable model for the real-life signal/sequence being studied.

If you don’t expect to encounter Markov chains, explain why not.

The following exercises are for self study and group discussion:

1. There are many additional problems in Stark and Woods that exercise the concepts developed in Chapter 6. You are strongly encouraged to work on them in your study group. You may find it advantageous to do it this week, while you are learning the material, rather than close to the final exam.

2. Review Chapter 7 from Stark and Woods.