Submit your solutions to the following problems:

1. Problem 1.8 from Stark and Woods.

2. Problem 1.13 from Stark and Woods.

3. Let $\Omega = (0, 1]$, and enumerate the elements of $\sigma(\mathcal{G})$, the smallest $\sigma$-field that contains the collection of subsets $\mathcal{G} = \left\{ (\frac{1}{3}, \frac{1}{2}), (\frac{1}{2}, 1] \right\}$.

4. Show that if a collection $\mathcal{F}$ of subsets of $\Omega$ is closed under complementation and countable unions, it is also closed under countable intersections.

5. Ross’s Paradox: Consider an infinite number of balls labeled $1, 2, 3, 4, \ldots$, an empty urn that can hold an infinite number of balls, and the following two experiments.

Exp #1. At time $t = 0$, balls $1, 2, \ldots, 10$ are added to the urn, and the ball labeled 1 is removed.

At time $t = \frac{1}{2}$, balls $11, 12, \ldots, 20$ are added to the urn, and the ball labeled 2 is removed.

At time $t = \frac{3}{4}$, balls $21, 22, \ldots, 30$ are added to the urn, and the ball labeled 3 is removed.

At time $t = \frac{7}{8}$, balls $31, 32, \ldots, 40$ are added to the urn, and the ball labeled 4 is removed,

and so on. Assume that the addition and removal of balls takes zero time.

$\Rightarrow$ Is the urn empty at $t = 1$? Note: For every $n$, the ball labeled $n$ is removed at some time $t < 1$.

Exp #2. At time $t = 0$, balls $1, 2, \ldots, 9$ are added to the urn, and the ball labeled 1 is relabeled 10.

At time $t = \frac{1}{2}$, balls $11, 12, \ldots, 19$ are added to the urn, and the ball labeled 2 is relabeled 20.

At time $t = \frac{3}{4}$, balls $21, 22, \ldots, 29$ are added to the urn, and the ball labeled 3 is relabeled 30.
At time $t = \frac{7}{8}$, balls 31, 32, ..., 39 are added to the urn, and the ball labeled 4 is relabeled 40, and so on. Assume that the addition and relabeling of balls takes zero time.

⇒ Is the urn empty at $t = 1$?

Reflect on the two experiments and write down your thoughts in 50-200 words. Be cogent and precise in your prose.

The following exercises are for self study and group discussion:

1. Read pages 1-14 from the Papamarcou notes.

2. Let $\omega$ be uniformly distributed on $\Omega = (0, 1]$. Consider two random processes

   $$X(t) = \begin{cases} 
   1 & \text{if } t = \omega, \\
   0 & \text{otherwise},
   \end{cases} \quad \text{and} \quad Y(t) = 0, \quad \text{where } t \in (0, 1].$$

   Discuss why all finite dimensional statistics of $X(t)$ and $Y(t)$ are identical, and yet

   $$P\left(\max_{0<t\leq1} X(t) \leq \frac{1}{2}\right) = 0 \quad \text{while} \quad P\left(\max_{0<t\leq1} Y(t) \leq \frac{1}{2}\right) = 1.$$

   This situation illustrates the complexity (impracticality?) of a complete specification of random processes.

3. After finishing Problem 5 above, you may discuss Ross’s paradox with each other.