ECE 520.651 Random Signal Analysis

Homework # 6

Due 9:00 AM on Tuesday, October 20, 2009.

Carefully review Section 18 from Prof. Papamarcou's notes (p65–73) and Section 6.7 from Stark and Woods (p375–383) before starting the homework.

1. Solve problem **6.41** from Stark and Woods.

   (d) In your answer to (c), point out where the conclusion fails if x is not a point of continuity of F_X.

2. It is obvious that if a random sequence $X_n(\omega)$ converges *pointwise*, then the limit is unique. i.e. if $X_n \to X$ pointwise and $X_n \to Y$ pointwise, then it is obvious that

$$X(\omega) = Y(\omega) \qquad \text{for every } \omega \in \Omega.$$

   Show that the limit of almost sure convergence is *essentially* unique. i.e., show that if $X_n \to X$ a.s. and $X_n \to Y$ a.s., then

$$P(\{\omega \in \Omega : X(\omega) = Y(\omega)\}) = 1.$$

3. Is the limit *essentially* unique for convergence in probability as well? i.e. if $X_n \to X$ in probability and $X_n \to Y$ in probability, then is it true that

$$P(\{\omega \in \Omega : X(\omega) = Y(\omega)\}) = 1?$$

4. *The Borel-Cantelli Lemmas*: Let A_n, $n = 1, 2, \ldots$, be a sequence of events in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let

$$A = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

   represent the event that infinitely many of the A_n occur. Show that

   (a) if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(A) = 0$;

   (b) if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and the A_n are mutually independent, then $\mathbb{P}(A) = 1$.

   Hints: $P(E \cup F) \leq P(E) + P(F)$ and more generally $P(\cup_n E_n) \leq \sum_n P(E_n)$, while for independent events, $P(E \cap F) = P(E)P(F)$ and more generally $P(\cap_n E_n) = \prod_n P(E_n)$.

5. We have shown in class that convergence in r-th mean does not, in general, imply almost sure convergence. Here, we will show that if X_n converges in the r-th mean *sufficiently fast*, then it *does* converge almost surely.

   Specifically, show that if $\sum_{n=1}^{\infty} E\left[|X_n - X|^r\right] < \infty$ for some $r > 0$, then $X_n \to X$ a.s.