ECE 520.651 Random Signal Analysis

Homework # 5

Due 9:00 AM on Tuesday, October 14, 2007.

Carefully review Section 18 from Prof. Papamarcou's notes (p65–73) and Section 6.7 from Stark and Woods (p375–383) before starting the homework.

1. Solve problem **3.30** from Stark and Woods.

2. Solve problem **6.36** from Stark and Woods.

3. Solve problem **6.41** from Stark and Woods.

   (d) In your answer to (c), point out where the conclusion fails if x is not a point of continuity of F_X.

4. Let X_n, $n = 1, 2, \ldots$, and X be random variables on a common underlying probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$, and consider the sequence of events $A_n^\epsilon = \{\omega : |X_n(\omega) - X(\omega)| > \epsilon\}$.

   Show that if $\sum_{n=1}^{\infty} \mathbb{P}(A_n^\epsilon) < \infty$ for all $\epsilon > 0$, then $X_n \to X$ almost surely.

5. *The Borel-Cantelli Lemmas*: Let A_n, $n = 1, 2, \ldots$, be a sequence of events in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let

$$A = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

   represent the event that infinitely many of the A_n occur. Show that

   (a) if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(A) = 0$;

   (b) if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and the A_n are mutually independent, then $\mathbb{P}(A) = 1$;

   (c) the statement above is false if the independence assumption is dropped;

   (d) if $\sum_{n=1}^{\infty} E\left[|X_n - X|^r\right] < \infty$ for some $r > 0$, then $X_n \to X$ almost surely.

   Therefore, if X_n converges in the r-th mean *sufficiently fast*, then it converges a. s.

   Hints: $P(E \cup F) \leq P(E) + P(F)$ and more generally $P(\cup_n E_n) \leq \sum_n P(E_n)$, while for independent events, $P(E \cap F) = P(E)P(F)$ and more generally $P(\cap_n E_n) = \prod_n P(E_n)$.

Review Sections 5.1–5.3 and 5.7 after you finish the homework. We will resume with the remainder of Chapter 5 after finishing convergence of random sequences.