Read these instructions before starting the examination.

(i) This is an open-book examination. Use of any two textbooks, Prof Papamarcou’s notes and one 8.5” × 11” sheet of paper with formulae written in your own hand is permitted.

Photocopied material from additional books, class notes or homework solutions, material obtained via the Internet etc. are not permitted.

(ii) Use of electronic calculators is not permitted; please put them away.

(iii) Show all your work clearly. Points may be deducted for illegible or unclear answers.

(iv) Write your answers in the space provided. Use the unprinted side of the pages if needed.

(v) There are five questions for a total of 100 points. You don’t have to answer them sequentially; answer what you find easiest first, and what you find hardest last. Use the check-list below to make sure you answer all the questions.

Best of luck!

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TOTAL /100 Points
Question No. 1: Analysis of Random Sequences.

Let a collection of sequences $x[n; \theta_k]$ be given in terms of a deterministic parameter $\theta_k$ as

$$
\left\{ x[n; \theta_k] = \cos \left( \frac{2\pi n}{5} + \theta_k \right) \right\}, \quad k = 0, 1, \ldots, N - 1.
$$

Now, let $\Theta$ be a uniformly distributed random variable on the set $\{\theta_0, \theta_1, \ldots, \theta_{N-1}\}$, and set

$$
X[n] = \cos \left( \frac{2\pi n}{5} + \Theta \right), \quad n = 0, 1, 2, \ldots
$$

(a) Is $X[n]$ a random sequence? If so, describe the underlying probability space $(\Omega, \mathcal{F}, P)$, and the mapping $X : \Omega \to \mathbb{R}^N$, discuss the range of this mapping. If not, explain why.

(b) Compute $E[X[n]]$ when $\theta_k = \frac{2\pi k}{N}$, $k = 0, 1, \ldots, N - 1$.

(c) For the same $\theta_k$ as in part (b) above, compute $E[X[m]X[n]]$. Assume $N > 2$ if needed.
Extra work-space 1
Question No 2: Markov Random Sequences.

Consider a discrete-time Markov random sequence given by

\[ X[n] = r X[n-1] + Z[n], \]

where \( Z[n] \) is white noise with variance \( \sigma_Z^2 \). (cf. Stark and Woods, p 436, for a definition.)

(a) First, view \( X[n] \) as the output of a linear system whose input is \( Z[n] \). Assume further that \( |r| < 1 \) and that the system has been running for a long time, i.e. \( -\infty < n < +\infty \).

Find the power spectral density \( S_{XX}(\omega) \) of \( X[n] \).

Find the autocorrelation function \( R_{XX}(n) \) of \( X[n] \).

(b) Now, let \( n = 1, 2, \ldots \), and view \( X[n] \) as a random walk beginning at \( X[0] = 0 \). Assume further that the \( Z[n] \) are independent and identically distributed with zero mean.

Show that the subsampled sequence \( Y[n] = X[2n] \) is a Markov random sequence.

Determine the variance function \( \sigma_Y^2[n] = E[|Y[n] - \mu_Y[n]|^2] \) for \( n > 0 \).
Extra work-space 2
Question No 3: The Poisson Counting Process (Stark and Woods 7.8).

This problem concerns the construction of the Poisson counting process \( X(t) \) as given in Stark and Woods, Section 7.2 p 408-410.

(a) Derive Equation (6.1-15), i.e. show that the density of the \( n \)-th arrival epoch \( T[n] \) is

\[
f_{T}(t; n) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} u(t), \quad n > 0.
\]

In the derivation of the property that the increments of the process are Poisson distributed, i.e.,

\[
P(X(t_a) - X(t_b) = n) = \frac{[\lambda(t_a - t_b)]^n}{n!} e^{-\lambda(t_a - t_b)} u(n), \quad t_a > t_b,
\]

it is implicitly assumed that the time from the beginning of any interval \((t_b, t_a]\) to the first arrival in that interval is exponentially distributed. Actually, this is not obvious, because this time is only part of the interarrival time between the last arrival before \( t_b \) and the first arrival in \((t_b, t_a]\).

Let \( \tau'[i] = T[i] - t_b \) be the random variable that denotes the residual interarrival time after \( t_b \), as opposed to the (full) interarrival time \( \tau[i] = T[i] - T[i-1] \). Then,

\[
\tau'[i] = \tau[i] - T,
\]

where the random variable \( T = t_b - T[i-1] \) is the elapsed interarrival time before \( t_b \).

This is illustrated in Figure P7.8 on p 473 in the Stark and Woods book, where the last arrival epoch before \( t_b \) is designated \( t_{i-1} \) and the first arrival epoch in \((t_b, t_a]\) is \( t_i \).

(b) Find the conditional CDF of \( \tau'[i] \) given \( T = t \):

\[
F_{\tau'[i]}(\tau' \mid T = t) = P(\tau'[i] \leq \tau' \mid T = t) = P(\tau[i] \leq \tau' + t \mid \tau[i] \geq t).
\]

(c) Compute the unconditional CDF of \( \tau'[i] \) from the answer to part (b).

Due to the preceding properties, the exponential distribution is said to be memoryless. It is the only continuous distribution with this property.
Extra work-space 3
Question No 4: Estimation of Nonrandom Parameters (Homework #11.3).

Let \( \theta \in (0, \infty) \) be a fixed but unknown parameter in a parametric family of densities

\[
f_{\theta}(y) = \frac{1}{\theta} \exp \left\{ -\frac{y}{\theta} \right\} u(y), \quad y \in \mathbb{R},
\]

and let \( Y_1^n \equiv Y_1, \ldots, Y_n \), be i.i.d. with common density \( f_{\theta}(\cdot) \). Consider the problem of minimizing the squared error in the estimation of \( \theta \) from \( Y_1^n \).

(a) Compute the Cramér-Rao lower bound for unbiased estimators of \( \theta \).

(b) Compute the maximum likelihood estimator \( \hat{\theta}_{ML}(Y_1^n) \).

(c) Show that \( \hat{\theta}_{ML}(Y_1^n) \) is the MVUE by computing its bias and variance.

(d) Compute the mean squared error of the estimator \( \tilde{\theta}(Y_1^n) = \frac{1}{n+1} \sum_{k=1}^n Y_k \).

(e) Discuss the bias-variance tradeoff by comparing \( \hat{\theta}_{ML}(Y_1^n) \) and \( \tilde{\theta}(Y_1^n) \).

Explain the basic intuition using \( n = 1 \), then discuss larger \( n \).
Extra work-space 4
Question No 5: Estimation of Random Variables (Poor IV.F.11).

Suppose that we observe a sequence

\[ Y_k = X_k + N_k, \quad k = 1, 2, \ldots, \]

where \( N_1, N_2, \ldots, \) are independent and identically distributed Gaussian random variables with zero mean and common variance \( \sigma^2 \), and \( X_1, X_2, \ldots, \) are defined by the equations

\[ X_k = \alpha X_{k-1} \quad k = 1, 2, \ldots, \begin{align*}
\text{beginning with} \quad X_0 &= \Theta, \end{align*} \]

where \( \alpha > 0 \) is a known constant and \( \Theta \) is a Gaussian random variable with zero mean and variance \( q^2 \). We are interested in estimating \( \Theta \) from the observed sequence.

(a) Assuming that \( \Theta \) is independent of all the \( N_k \)'s, find the MMSE estimator \( \hat{\Theta}_n \) of \( \Theta \) based on \( Y_1, Y_2, \ldots, Y_n \). Comment on the computational complexity of this estimator.

(b) For each \( k = 1, 2, \ldots \), let \( \hat{\Theta}_k \) denote the MMSE estimator of \( \Theta \) based on \( Y_1, Y_2, \ldots, Y_k \). Show that \( \hat{\Theta}_k \) can be computed recursively as

\[ \hat{\Theta}_k = \frac{1}{G_k} \left[ G_{k-1} \hat{\Theta}_{k-1} + \alpha^k Y_k \right], \quad k = 1, 2, \ldots, \]

where \( \hat{\Theta}_0 = 0 \) and the coefficients \( G_k \) are also recursively computed as

\[ G_k = G_{k-1} + \alpha^{2k}, \quad k = 1, 2, \ldots, \begin{align*}
\text{beginning with} \quad G_0 &= \frac{\sigma^2}{q^2}. \end{align*} \]

What is the computational complexity of \( \hat{\Theta}_n \) based on these recursions?

(c) Find an expression for the mean squared error \( E \left[ (\Theta - \hat{\Theta}_n)^2 \right] \).

Discuss what happens when \( n \to \infty; \ q^2 \to \infty; \ \sigma^2 \to \infty; \ \alpha < 1; \ \alpha = 1; \ \alpha > 1. \)
Extra work-space 5
Extra work-space 6