Read these instructions before starting the examination.

(i) This is an open-book examination. Use of any two textbooks, Prof Papamarcou’s notes and one 8.5 × 11 sheet of paper (“crib sheet”) with formulae written in your own hand is permitted. Photocopied material from additional books, class notes or homework solutions, material obtained via the Internet etc. are not permitted.

(ii) Use of electronic calculators is permitted for numeric calculations only.

(iii) Show all your work clearly. Points may be deducted for illegible or unclear answers.

(iv) Write your answers in the space provided. Use the unprinted side of the pages if needed.

(v) There are five questions for a total of 100 points, and a sixth bonus question. Use the check-list below to keep track of your progress.

Best of luck!

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TOTAL /100 Points
Question No 1: Polya’s Urns. An urn contains $b$ black and $r$ red balls. A ball is drawn at random. It is replaced and, moreover, $c$ balls of the color drawn and $d$ balls of the opposite color are added to the urn. A new random drawing is made from the urn (now containing $b+r+c+d$ balls), and this procedure is repeated.

(a) Describe phenomena modeled by $(c = 0, d = 0)$ and $(c = −1, d = 0)$.

(b) If $c > 0$ and $d = 0$, we get a model of phenomena such as contagious diseases.

What is the probability that the ball at the second drawing is black?

What is the probability that the ball at the $n$-th drawing, $n = 1, 2, \ldots$, is black?

Caution: The number of black balls in the urn after $n − 1$ drawings is a random variable. Proofs that treat it like a deterministic quantity will incur negative marks.

(c) If $c = −1$ and $d = 1$ we get a model of diffusion.

Consider a conceptual experiment in which $N = b + r$ gas molecules are distributed in two connected containers, with $b$ representing the number of molecules in the first container and $r$ in second. At each time-step, a single randomly chosen molecule diffuses from the container it was in to the other container.

Does the number of molecules in the first container (equivalently, the number of black balls in the urn) form a Markov chain?

If so, draw a state transition diagram and write down the transition probability matrix. If not, illustrate the violation of the Markov property.
Extra work-space 1
Question No 2: Second Order Characterization of Processes. Let $X[n]$ be a real-valued stationary random sequence with $E\{X[n]\} = \mu_X$ and $E\{X[n+m]X[n]\} = R_{XX}[m]$. If $X[n]$ is the input to a D/A converter, the continuous-time output can be idealized as the analog random process $X_a(t)$ with

$$X_a(t) = X[n], \quad \text{for} \quad n \leq t < n+1, \quad \forall \ t \in \mathbb{R}.$$  

(a) Find the mean function $E\{X_a(t)\} = \mu_{X_a}(t)$ as a function of $\mu_X$.

(b) Find the autocorrelation function $E\{X_a(t_1)X_a(t_2)\} = R_{X_aX_a}(t_1, t_2)$ in terms of $R_{XX}[m]$.

(c) Is the analog signal $X_a(t)$ wide-sense stationary? Elaborate.

Note: To write your answers compactly, you may wish to use the notation $\lfloor t \rfloor$ to denote the largest integer that is not larger than $t$. 

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Extra work-space 2
Question No 3: *Martingales and MMSE Estimators.* Let $\Theta \sim w(\theta)$ be a random variable and given $\Theta = \theta$, let $Y_0, Y_1, \ldots$, be i.i.d. random variables with common density $f_\theta(y)$. Recall that the MMSE estimate of $\Theta$ from $Y_0, Y_1, \ldots, Y_n$ is the conditional mean,

$$\hat{\Theta}_n \overset{\Delta}{=} E[\Theta | Y_0, Y_1, \ldots, Y_n].$$

(a) Show that $\hat{\Theta}_n$ is a Martingale.

(b) Show that if $\Theta$ has finite variance under $w(\theta)$, then $\hat{\Theta}_n$ converges almost surely.

(c) What is $\lim_{n \to \infty} \hat{\Theta}_n$?
Extra work-space 3
Question No 4: Channel Equalization. Let $X(t)$ and $N(t)$, $t \in \mathbb{R}$, be orthogonal, zero-mean, jointly wide-sense stationary random processes. $X(t)$ represents the input signal to a linear time-invariant (LTI) channel with impulse response $h(t)$, and $N(t)$ is an additive noise:

$$Y(t) = h(t) \ast X(t) + N(t).$$

We wish to design a LTI filter with impulse response $g(t)$ to process $Y(t)$, so that

$$\hat{X}(t) = g(t) \ast Y(t) \quad \text{statistically minimizes} \quad \Delta(t) = \hat{X}(t) - X(t),$$

the error in the reconstruction of the signal $X(t)$. For instance, we may wish to minimize $E[\Delta^2(t)] = R_{\Delta\Delta}(0) = \frac{1}{2\pi} \int S_{\Delta\Delta}(\omega) d\omega$. To this end, we wish to compute the power spectral density of the reconstruction error.

(a) Write $S_{YY}(\omega)$ in terms of $S_{XX}(\omega)$, $S_{NN}(\omega)$ and $H(\omega)$.

(b) Write $S_{\hat{X}X}(\omega)$ in terms of $S_{XX}(\omega)$, $S_{NN}(\omega)$, $H(\omega)$ and $G(\omega)$.

(c) Write $S_{\Delta\Delta}(\omega)$ in terms of $S_{XX}(\omega)$, $S_{NN}(\omega)$, $H(\omega)$ and $G(\omega)$.

Describe the optimal filter $g(t)$ or, equivalently, its frequency response $G(\omega)$. 

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Extra work-space 4
Question No 5: Parameter Estimation. Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. random variables distributed uniformly on $[a, b]$.

(a) Find the MLE of $\theta = (a, b) \in \mathbb{R}^2$.

(b) Determine if the MLE is unbiased; if not, compute its bias.

(c) Find a sufficient statistic for estimating $\theta$ from $Y_1, \ldots, Y_n$.

(d) Find an unbiased estimator of $\theta$. If it is not already a function of the sufficient statistic, use the Rao-Blackwell Theorem to improve the variance of your unbiased estimator.
Extra work-space 5
Question No 6: Linear Estimation.

(a) Let $X \sim \mathcal{N}(0, \sigma^2)$ and $W \sim \text{Laplace}(\lambda)$ be independent, and let $Y = X^3 + W$. Find the LMMSE estimate of $X$ based on $Y$.

(b) Let $X$ and $Y$ be random vectors with known means and covariance matrices. Do not assume zero means. Find the best purely linear estimator of $X$ based on $Y$; i.e. find the matrix $A$ that minimizes $E[\|X - AY\|^2]$.

(c) For the same setting as (b) above, find the best constant estimator of $X$; i.e. find the vector $b$ that minimizes $E[\|X - b\|^2]$. 
Extra work-space 6
Extra work-space 7