ECE 520.651 Random Signal Analysis

First (Mid-Term) Examination, Fall 2006

5:00 PM — 8:00 PM, October 26, 2006.

Name: ____

Read these instructions before starting the examination.

(i) This is an open-book examination. Use of any one textbook and Prof Papamarcou's notes is permitted. Photocopied material from additional books, class notes or homework solutions, material obtained via the Internet *etc.* are **not** permitted.

(ii) Use of electronic calculators is permitted for numeric calculations only.

(iii) Show all your work clearly and concisely. Points may be deducted for illegible or unclear answers.

(iv) Write your answers in the space provided. Use the unprinted side of the pages for additional space.

(v) There are five mandatory questions for a total of 100 points. Use the check-list below to keep track of your progress.

Best of luck!

Question № 1 (a) (b)	/20 Points
Question № 2 (a) (b)	/20 Points
Question № 3 (a) (b) (c)	/20 Points
Question № 4 (a) (b)	/20 Points
Question № 5 (a) (b)	/20 Points
TOTAL	/100 Points

Extra work-space 1

Question No 1: *Sample Spaces, Event Spaces and Random Variables.*

(1a) Show that for any mapping $X : \Omega \to \mathbb{R}$, inverse images under X preserve the following set operations. (12 points)

(i) *Complementation*: if $A \subset \mathbb{R}$, then $X^{-1}(A^c) = [X^{-1}(A)]^c$.

(ii) *Countable Unions*: if $A_1, A_2, \ldots \subset \mathbb{R}$, then $X^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} X^{-1}(A_i)$.

(iii) *Countable Intersections*: if $A_1, A_2, \ldots \subset \mathbb{R}$, then $X^{-1}\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcap_{i=1}^{\infty} X^{-1}(A_i)$.

(1b) Let $X : \Omega \to \mathbb{R}$ be a mapping that takes only n distinct values, say, $x_1, x_2, \ldots, x_n$. Describe the smallest σ-field on Ω for which X is a random variable. (8 points)

Question No 2 *Independent Events and Conditional Independence.*

(2a) Let $\Omega = [0,1)$, $\mathcal{F} = \mathcal{B}([0,1))$ and P be the Lebesgue measure. Determine whether the following events are mutually independent: (8 points)

$$\begin{aligned} A &= \left[0, \frac{1}{2}\right) \\ B &= \left[0, \frac{1}{4}\right) \bigcup \left[\frac{1}{2}, \frac{3}{4}\right) \\ C &= \left[0, \frac{1}{8}\right) \bigcup \left[\frac{1}{4}, \frac{3}{8}\right) \bigcup \left[\frac{1}{2}, \frac{5}{8}\right) \bigcup \left[\frac{3}{4}, \frac{7}{8}\right) \end{aligned}$$

(2b) For any three events A, B and C, show that (12 points)

$$P(A \cap C|B) = P(A|B) \times P(C|B) \quad \text{if and only if} \quad P(A|B \cap C) = P(A|B).$$

In this case, we say that A and C are *conditionally independent* given B.

(i) Is independence of A and C *sufficient* to make them conditionally independent?

(ii) Is independence *necessary* for them to be conditionally independent?

Question No̲ 3: *Convergence of Random Variables.* Let X_n be a sequence of independent Cauchy$\left(\frac{1}{n}\right)$ random variables.

(3a) Show that X_n converges in probability. (7 points)

(3b) Does X_n converge in any other sense? (8 points)

(3c) If $g : \mathbb{R} \to \mathbb{R}$ is a bounded and nonnegative function satisfying $\lim_{x \to 0} g(x) = 0$, show that $Y_n = g(X_n)$ converges in the r-th mean. (7 points)

Extra work-space 2

Question No̲ 4: *Computing Probability Densities of Functions of Random Variables.*

(4a) Let $Z = XY$, where X and Y are continuous random variables with joint probability density function $f_{XY}(x, y)$. Obtain the probability density function $f_Z(z)$ in terms of $f_{XY}(\cdot, \cdot)$. (10 points)

(4b) Let X and Y be independent $\mathcal{N}(0,1)$ random variables, and let U and V be independent of X and Y. Show that

$$Z = \frac{UX + VY}{\sqrt{U^2 + V^2}}$$

is also distributed $\mathcal{N}(0,1)$. (10 points)

Question No̲ 5: *Expectations and Characteristic Functions of Random Variables.*

(5a) Let X be a $\mathbb{R}$-valued random variable with pdf $e^{-x}u(x)$ and, given $X = x$, let Y be *conditionally* Gaussian: $\mathcal{N}(0, x^2)$. Compute $E[Y^2]$ and $E[X^2Y^2]$. (10 points)

(5b) Compute the characteristic function of $Y = X^2$ when $X \sim \mathcal{N}(\mu, \sigma^2)$. (10 points)