ECE 520.651 Random Signal Analysis

Second (Final) Examination, Fall 2006

9:00 AM — 12:00 PM, December 22, 2006.

Name: ____

Read these instructions before starting the examination.

(i) This is an open-book examination. Use of the Stark and Woods textbook, the Poor textbook and Prof. Papamarcou's notes, and the Kalman Filtering handout provided in class are permitted. No other material is permitted.

(ii) You will not need an electronic calculator for this exam!

(iii) Show all your work clearly and concisely. Points may be deducted for illegible or unclear answers.

(iv) Write your answers in the space provided. Use the unprinted side of the pages for additional space.

(v) There are five mandatory questions for a total of 100 points. Use the check-list below to keep track of your progress.

Best of luck!

Question № 1 (a) (b) (c) (d)		/20 Points
Question № 2 (a) (b) (c)		/20 Points
Question № 3 (a) (b) (c) (d)		/20 Points
Question № 4 (a) (b) (c) (d)		/20 Points
Question № 5 (a) (b) (c) (d)		/20 Points
Question № 6 (a) (b)	Bonus	/10 Points
TOTAL		/100 Points

Question No 1: *Markov Chains.* Let X_n, $n = 0, 1, 2, \ldots$, be a time-homogeneous Markov chain taking values in a discrete and finite set $\mathcal{S} = \{1, 2, \ldots, K\}$, with transition probability matrix $\mathbf{P}_{K \times K}$, where $\mathbf{P}_{ij} = \text{Prob}(X_n = j | X_{n-1} = i)$, $n = 1, 2, \ldots$, and initial distribution $\mathbf{p}[0]$, where $\mathbf{p}_i[0] = \text{Prob}(X_0 = i)$.

(1a) In general, is X_n wide sense stationary? Strictly stationary? (3 points)

(1b) For $K = 2$ and $\mathbf{P} = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$, with $0 < \alpha, \beta < 1$, answer the following.

1. Write $\mathbf{p}[n]$ in terms of $\mathbf{p}[0]$ and $\mathbf{P}$. (2 points)
2. Describe the behavior of $\mathbf{p}[n]$ as $n \to \infty$. If your answer depends on $\mathbf{p}[0]$, describe the different cases. (2 points)
3. Find the steady state distribution $\mathbf{p}^*$ of the Markov chain. (2 points)
4. In what sense, if any, does the sequence X_n converge? (4 points)

(1c) For any K, i.e. not necessarily $K = 2$, is there a choice of $\mathbf{p}[0]$ and $\mathbf{P}$ that could guarantee that X_n is wide sense stationary? Strictly stationary? (4 points)

(1d) If X_n is a wide sense stationary random sequence, i.e. not necessarily a Markov chain, in what sense does it converge? What if it is strictly stationary? (3 points)

Extra Workspace 1

Question No 2: *Random Sequences.* We are given a random sequence $X[n]$, $n = 0, 1, \ldots$, with conditional pdf's

$$f_{X[n]|X[n-1]}(x'|x) = \alpha \exp\{-\alpha(x' - x)\} \, u(x' - x), \qquad n = 1, 2, \ldots,$$

where $u(\cdot)$ is the unit step function and $X[0]$ has the pdf $f_{X[0]}(x) = \delta(x)$.

(2a) Find the marginal pdfs $f_{X[1]}(x)$ and $f_{X[2]}(x)$ of $X[1]$ and $X[2]$. (8 points)

(2b) Find the marginal pdf $f_{X[n]}(x)$ for arbitrary $X[n]$. (*Hint*: Try induction.) (8 points)

(2c) Is the random sequence $X[n]$ any of the following? (4 points)

1. A strictly stationary random sequence.
2. A wide sense stationary random sequence.
3. An independent increment random sequence.
4. A Markov chain.

Extra Workspace 2

Question N$\underline{o}$ 3: *Stationary Random Processes.* Consider the LSI system drawn on the board. Let $X(t)$ and $N(t)$ be zero-mean, wide sense stationary and mutually uncorrelated random processes, with power spectral densities $S_{XX}(\omega)$ and $S_{NN}(\omega)$ respectively.

(3a) Determine the power spectral density $S_{YY}(\omega)$ of the random process $Y(t)$. (5 points)

(3b) Determine the cross power spectral densities $S_{XY}(\omega)$ and $S_{YX}(\omega)$. (5 points)

(3c) Compute the power spectral density of $Z(t) = Y(t) - X(t)$. (5 points)

(3d) Assume that $h(t) = \alpha \times \delta(t)$ for some real constant α, and choose the value of α so as to minimize the mean squared error $E\left[Z^2(t)\right]$ in the reconstruction of $X(t)$ from $Y(t)$. (*Hint*: Recall that for a WSS process, $E\left[Z^2(t)\right] = R_{ZZ}(0)$.) (5 points)

Extra Workspace 3

Question N$\underline{\text{o}}$ 4: *Linear Minumum Mean Squared Error (LMMSE) Estimation.*

(4a) Use the orthogonality principle to show that the MMSE (6 points)

$$\varepsilon^2 = E\left[(X - E[X|Y])^2\right],$$

for real valued random variables can be expressed as

$$\varepsilon^2 = E\left[X(X - E[X|Y])\right]$$

or as

$$\varepsilon^2 = E\left[X^2\right] - E\left[(E[X|Y])^2\right].$$

(4b) Generalize to the case where $\mathbf{X}$ and $\mathbf{Y}$ are real-valued random vectors, i.e. show that the MMSE matrix is (4 points)

$$\begin{aligned}\varepsilon^2 &= E\left[(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^{\mathrm{T}}\right] \\ &= E\left[\mathbf{X}(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^{\mathrm{T}}\right] \\ &= E\left[\mathbf{X}\mathbf{X}^{\mathrm{T}}\right] - E\left[E[\mathbf{X}|\mathbf{Y}](E[\mathbf{X}|\mathbf{Y}])^{\mathrm{T}}\right].\end{aligned}$$

Modify Theorem 9.1-3 in Stark and Woods pages 560-562 to specify the LMMSE estimate of the zero-mean random sequence $X[n]$ based upon *only the* $p+1$ *most recent observations* of the zero-mean random sequence $Y[n]$, i.e.

$$\widehat{E}\left[\mathbf{X}[n]\,|\mathbf{Y}[n],\mathbf{Y}[n-1],\ldots,\mathbf{Y}[n-p]\right]=\sum_{i=0}^{p}\tilde{a}_i\mathbf{Y}[n-i].$$

(4c) Write equations analogous to Equations (9.1-25) and (9.1-26). (5 points)

(4d) Derive an equation for $\varepsilon^2_{\min}$ corresponding to Equation (9.1-27). (5 points)

Question No̲ 5: *Parameter Estimation.* Suppose we observe a sequence $Y_1, Y_2, \ldots, Y_n$, given by

$$Y_k = N_k + \theta s_k, \qquad k = 1, 2, \ldots, n,$$

where $\underline{N} = [N_1 \ N_2 \ \ldots \ N_n]^{\mathrm{T}}$ is a zero-mean Gaussian random vector with positive definite covariance matrix Σ, $\underline{s} = [s_1 \ s_2 \ \ldots \ s_n]^{\mathrm{T}}$ is a known signal sequence, and θ is a fixed but unknown parameter.

(5a) Find the maximum likelihood estimate $\hat{\theta}_{\mathrm{ML}}(\underline{Y})$ of θ from $\underline{Y} = [Y_1 \ \ldots \ Y_n]^{\mathrm{T}}$. (5 points)

(5b) Compute the bias and variance of $\hat{\theta}_{\mathrm{ML}}(\underline{Y})$. (5 points)

(5c) Compute the Cramér-Rao lower bound for unbiased estimates of θ from $\underline{Y}$. (5 points)

(5d) Compare your answers in parts (5b) and (5c); if they are different, explain why, and if not then explain why the MLE achieves the CRLB. (5 points)

Extra Workspace 4

Question No̲ 6 *Kalman Filtering.* Consider the linear stochastic model

$$\begin{aligned} \underline{X}_{n+1} &= \mathbf{F}_n \underline{X}_n + \mathbf{G}_n \underline{U}_n, \qquad n = 0, 1, \ldots, \\ \underline{Y}_n &= \mathbf{H}_n \underline{X}_n + \underline{V}_n, \qquad n = 0, 1, \ldots, \end{aligned}$$

where, for each $n \geq 0$, $\underline{X}_n$, $\underline{U}_n$, $\underline{Y}_n$ and $\underline{V}_n$ are random vectors of dimension m, s, k and k respectively, $\mathbf{F}_n$, $\mathbf{G}_n$ and $\mathbf{H}_n$ are known matrices of appropriate dimensions, $\underline{U}_n$ and $\underline{V}_n$ are mutually uncorrelated and zero-mean, i.e.

$$E\left[\underline{U}_m \underline{U}_n^{\mathrm{T}}\right] = \mathbf{Q}_n \delta(n-m), \quad E\left[\underline{V}_m \underline{V}_n^{\mathrm{T}}\right] = \mathbf{R}_n \delta(n-m) \quad \text{and} \quad E\left[\underline{U}_m \underline{V}_n^{\mathrm{T}}\right] = \mathbf{0} \quad \forall\, m, n,$$

and uncorrelated with $\underline{X}_0$, and where $\underline{X}_0$ has mean m_0 and covariance Σ_0. Apart from these assumptions, the statistics of various random variables are arbitrary. e.g. no Gaussian assumptions are made.
Starting with $\hat{\underline{X}}_{0|-1} = m_0$, let us recursively define the quantities

$$\hat{\underline{X}}_{t|t} \triangleq \hat{\underline{X}}_{t|t-1} + \mathbf{K}_t(\underline{Y}_t - \mathbf{H}_t \hat{\underline{X}}_{t|t-1}) \qquad t = 0, 1, \ldots, \tag{1}$$

$$\text{and} \quad \hat{\underline{X}}_{t+1|t} \triangleq \mathbf{F}_t \hat{\underline{X}}_{t|t} \qquad t = 0, 1, \ldots, \tag{2}$$

where the matrix $\mathbf{K}_t$ is as defined[1] in Equation (V.B.15) in Poor, p210.

(6a) Show directly that if we view $\hat{\underline{X}}_{t|t}$ as a *linear estimator* of $\underline{X}_t$ based on $\underline{Y}_0, \ldots, \underline{Y}_t$, then the estimation error is orthogonal to the observations. i.e.

$$E\left\{(\underline{X}_t - \hat{\underline{X}}_{t|t})\underline{Y}_k^{\mathrm{T}}\right\} = \mathbf{0}, \qquad 0 \leq k \leq t.$$

Similarly, if we view $\hat{\underline{X}}_{t|t-1}$ as a *linear estimator* of $\underline{X}_t$ based on $\underline{Y}_0, \ldots, \underline{Y}_{t-1}$ then

$$E\left\{(\underline{X}_t - \hat{\underline{X}}_{t|t-1})\underline{Y}_k^{\mathrm{T}}\right\} = \mathbf{0}, \qquad 0 \leq k \leq t-1.$$

(NB: Showing directly means that we do not assume, as we did in the Gaussian case, that $\hat{\underline{X}}_{t|t} = E\left\{\underline{X}_t | \underline{Y}_0, \ldots, \underline{Y}_t\right\}$, etc. $\hat{\underline{X}}_{t|t}$ is simply as defined above.) (7 points)

(6b) Use the results of (6a) to argue that Equation (1) above, together with (2), is *the* LMMSE filter for estimating $\underline{X}_t$ from $\underline{Y}_0, \ldots, \underline{Y}_t$. (3 points)

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[1] Note that the computation of $\mathbf{K}_t$ involves only *known* quantities — Σ_0, $\mathbf{H}_0, \ldots, \mathbf{H}_t$, $\mathbf{R}_0, \ldots, \mathbf{R}_t$, $\mathbf{F}_0, \ldots, \mathbf{F}_{t-1}$, and $\mathbf{G}_0, \ldots, \mathbf{G}_{t-1}$ — as made explicit in Equations (V.B.16a) and (V.B.16b).

Extra Workspace 5