ECE 520.651
Random Signals Analysis

Midterm Examination #1, Take II

3:00 — 4:45 AM, December 10, 2003.

Name: _____

Read these instructions before starting the examination.

(i) This is an open-book examination. Use of the Stark and Woods textbook is permitted. Photocopied material from other books, handwritten/class notes, homework soultions, *etc.* are **not** permitted.

(ii) Use of electronic calculators is permitted for numeric calculations only.

(iii) Show all your work clearly and concisely. Points may be deducted for illegible or unclear answers.

(iv) Provide answers in the space provided. Use the unprinted side of the pages for additional space.

(v) There are four mandatory questions for a total of 80 points.

Best of luck!

Question No 1	/20 Points
Question No 2	/20 Points
Question No 3	/20 Points
Question No 4	/20 Points
TOTAL	/80 Points

Question No 1: Let $\{\Omega, \mathcal{F}, P\}$ be a probability space, and let $X : \Omega \to \mathbb{R}$ be real-valued random variable wrt the (usual) Borel σ-field on $\mathbb{R}$.

(1a) Show from first principles that the following are *bona fide* random variables.

$$X^+ = \max\{0, X\}, \qquad X^- = -\min\{0, X\}, \qquad |X| = X^+ + X^-, \qquad -X\,.$$

(Hint: Since X is a random variable, it follws that the inverse image of $(-\infty, a] \subset \mathbb{R}$ under X is in the σ-field $\mathcal{F}$ for every $a \in \mathbb{R}$. What happens under X^+ etc?)

(1b) Write the cumulative distribution function (cdf) of each of the four random variables in (1a) in terms of the cdf $F_X(x)$ of X.

Question No 2

(2a) Let X be a $\mathbb{R}$-valued random variable with probability density function

$$f_X(x) = \begin{cases} c\, e^{-2x} & \text{if } x \geq 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Determine the value of the constant c.

(ii) For $x > 0$ and $a > 0$, find $P[X \geq x + a]$.

(iii) For $x > 0$ and $a > 0$, find $P[X \geq x + a \,|X \geq a]$.

(2b) Consider a sequence of independent and identically distributed Gaussian random variables $X_1, X_2, \ldots, X_n$, each with mean μ, and variance σ^2. Let

$$S_n = X_1 + X_2 + \ldots + X_n.$$

(i) Find the probability density function $f_{S_n}(x)$ of S_n.

(ii) Find the probability density function of $M_n = \frac{1}{n} S_n$.

(iii) Find the probability density function of $C_n = \frac{1}{\sqrt{n}} S_n$.

Comment on S_n, M_n and C_n as $n \to \infty$.

(Hint: Linear transforms of Gaussian random vectors are Gaussian.)

Question No 3: Consider two independent real-valued random variables X and Y, and define $Z = \min\{X, Y\}$.

(3a) Compute $f_Z(z)$ when $f_X(x) = \alpha \exp[-\alpha x]\, u(x)$ and $f_Y(y) = \beta \exp[-\beta y]\, u(y)$.

(3b) Sketch $f_Z(z)$ when X and Y are both uniformly distributed on $[0, 1]$.

(3c) Compute the pdf $f_Z(z)$ (or the cdf $F_Z(z)$) more generally in terms of the pdf's $f_X(\cdot)$ and $f_Y(\cdot)$ (or the cdf's $F_X(\cdot)$ and $F_Y(\cdot)$).

Question No 4: Manipulation of random vectors.

(4a) Let $\mathbf{X} = [\, X_1 \quad X_2 \quad X_3 \,]^T$ be a random vector with mean $\mu = [\, 5 \quad -5 \quad 6 \,]^T$ and covariance matrix

$$\mathbf{K} = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 4 \end{bmatrix}.$$

Compute the mean and covariance of

$$Y = \mathbf{A}^T\mathbf{X} + B$$

for $\mathbf{A} = [\, 2 \quad -1 \quad 2 \,]^T$ and $B = 5$.

(4b) Let $\mathbf{X} = [\, X_1 \quad X_2 \quad X_3 \quad X_4 \,]^T$ be a zero-mean Gaussian random vector with covariance matrix $\mathbf{K} = \{K_{ij}\}_{4\times 4}$. Show that

$$E\left[X_1 X_2 X_3 X_4\right] = K_{12}K_{34} + K_{13}K_{24} + K_{14}K_{23}\,.$$