ECE 520.447
Introduction to Information Theory and Coding

Midterm Examination #1

9:00 — 9:50 AM, October 16, 2002.

Name: ____

Read these instructions before starting the examination.

(i) This is a open-book examination. Use of any one textbook is permitted. Photocopied material from other books, handwritten/class notes, *etc.* are not permitted.

(ii) Use of electronic calculators is permitted for numeric calculations only.

(iii) Show all your work clearly and concisely. Points may be deducted for illegible or unclear answers.

(iv) Provide answers in the space provided. Use the unprinted side of the pages in the examination booklet if necessary.

(v) There are three mandatory questions for a total of 50 points.

Best of luck!

Question № 1	/15 Points
Question № 2	/15 Points
Question № 3	/20 Points
TOTAL	/50 Points

Question No 1: Let $p(x, y, z)$ denote the joint pmf of random variables X, Y and Z taking values in $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$.

(1a) Show that

$$H(X,Y,Z) - H(X,Y) \leq H(X,Z) - H(X)$$

and find the condition(s) for equality. (5 points)

(1b) Show that

$$I(X;Y|Z) \geq I(Y;Z|X) - I(Y;Z) + I(X;Y)$$

and find the condition(s) for equality. (5 points)

(1c) Give an example of $p(x, y, z)$ for which $I(X; Y|Z) < I(X; Y)$ and another for which $I(X; Y|Z) > I(X; Y)$. (5 points)

Question No̲ 2: Consider $\{X_n\}_{n=0}^{\infty}$, the random walk on integers described in Problem 10 in Chapter 4 of the Cover and Thomas book. Starting with $X_0 = 0$ and taking a unit step in either direction with probability $\frac{1}{2}$ to reach X_1, the walk then proceeds in unit steps with some *inertia*. Specifically, X_{n+1} is a step in the current direction with probability $\frac{9}{10}$ and a step in the reverse direction with probability $\frac{1}{10}$ as described in the problem.

(2a) Which of the following are true about the random walk?

- $P(X_n|X_0, X_1, \ldots, X_{n-1}) = P(X_n|X_{n-1})$ for $n = 2, 3, \ldots$. (2 points)
- $P(X_n|X_0, X_1, \ldots, X_{n-1}) = P(X_n|X_{n-2}, X_{n-1})$ for $n = 2, 3, \ldots$. (2 points)
- $P(X_n|X_0, X_1, \ldots, X_{n-1}) = P(X_n|X_{n-3}, X_{n-2}, X_{n-1})$ for $n = 3, 4, \ldots$. (2 points)

(2b) Determine $P(X_n = x_n|X_1 = x_1, \ldots, X_{n-1} = x_{n-1})$. (4 points)

Hint: While $P(X_n = x_n) > 0$ for all integers $x_n \in [-n, n]$, the conditional probability $P(X_n = x_n|X_1 = x_1, \ldots, X_{n-1} = x_{n-1})$ is nonzero only for $x_n = x_{n-1} \pm 1$.

(2c) Determine $H(X_1, X_2, \ldots, X_n)$ and the entropy rate $H(\mathcal{X})$. (5 points)

Question No 3: Consider a random variable X which takes values in $\mathcal{X} = \{a, b, c, d\}$ with probabilities $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12}\right)$.

(3a) Construct a binary Huffman code for X. (5 points)

(3b) Show that there exist two different sets of optimal length assignments for the codewords. (5 points)

(3c) In a given code C, if the codeword length $\ell(x)$ of a particular symbol x exceeds its

Shannon codeword length, i.e. $\ell(x) > \left\lceil \log \frac{1}{p(x)} \right\rceil$, can C be optimal for X? (5 points)

(3d) Construct a ternary Huffman code for X. (5 points)