ECE 520.447
Introduction to Information Theory and Coding

Homework #1
Due in class at 9:00 AM, September 16, 2002.

September 8, 2002

1. Let (X, Y) be random variables taking values in $\mathcal{X} \times \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are discrete finite sets of cardinality $|\mathcal{X}| = M$ and $|\mathcal{Y}| = N$. Let P_{XY} be the joint p.m.f. of (X, Y).
   (a) What are the maximum possible values of $H(X, Y)$, $H(X)$, $H(Y|X)$ and $H(X|Y)$?
   (b) Characterize the p.m.f. P^*_{XY} which achieves the maximum in each case.
   (c) Characterize the p.m.f. which yields $H(X) = H(X|Y)$.

2. Consider an urn containing 50 red, 30 blue and 20 green balls. Let $X_1, X_2, \ldots, X_{10}$ be a random sequence of 10 balls drawn from the urn *with replacement*, where $X_i \in \{\text{red}, \text{blue}, \text{green}\}$. Let $R = R(X_1, X_2, \ldots, X_{10})$ be the number of red balls, $B = B(X_1, X_2, \ldots, X_{10})$ the number of blue balls, and $G = G(X_1, X_2, \ldots, X_{10})$ the number of green balls in the sequence.
   (a) Write down the probability of observing a particular sequence "red, red, red, red, red, blue, blue, blue, green, green." Does this probability depend on this particular order of the drawing?
   (b) Write a general expression for the probability of a sequence with $R = r$, $B = b$, and $G = g$. Evaluate this expression for $R = 5$, $B = 3$, and $G = 2$.
   (c) How many sequences of drawings are there with $R = r$, $B = b$, and $G = g$? Evaluate this number for $R = 5$, $B = 3$, and $G = 2$.
   (d) What is the most likely value (are the most likely values) of the sequence $X_1, \ldots, X_{10}$? How many sequences are there with this highest probability?
   (e) What is the *total* probability of *all sequences* with $R = r$, $B = b$, and $G = g$? Evaluate this total probability for $R = 5$, $B = 3$, and $G = 2$, for $R = 4$, $B = 3$, and $G = 3$ and for $R = 10$, $B = 0$, and $G = 0$.

   When 10 balls are drawn with replacement from this urn, what outcome should one "expect?" Explain is one or two sentences.

3. **Chapter 2, Problem 13:** Suppose one has n coins, among which there may or may not be one counterfeit coin. If there is a counterfeit coin, it will be either heavier or lighter than the other coins. The coins are to be weighed by a balance.
   (a) Find an upper bound on the number of coins n so that k weighings will find the counterfeit coin (if any) and correctly declare it to be heavier or lighter.
   (b) What is the coin weighing strategy for $n = 12$ and $k = 3$ weighings?