ECE 520.447
Introduction to Information Theory and Coding

Homework #1
Due in class at 9:00 AM, September 16, 2002.

September 8, 2002

1. Let \((X,Y)\) be random variables taking values in \(\mathcal{X} \times \mathcal{Y}\), where \(\mathcal{X}\) and \(\mathcal{Y}\) are discrete finite sets of cardinality \(|\mathcal{X}| = M\) and \(|\mathcal{Y}| = N\). Let \(P_{XY}\) be the joint p.m.f. of \((X,Y)\).

(a) What are the maximum possible values of \(H(X,Y)\), \(H(X)\), \(H(Y|X)\) and \(H(X|Y)\)?

(b) Characterize the p.m.f. \(P^*_{XY}\) which achieves the maximum in each case.

(c) Characterize the p.m.f. which yields \(H(X) = H(X|Y)\).

2. Consider an urn containing 50 red, 30 blue and 20 green balls. Let \(X_1, X_2, \ldots, X_{10}\) be a random sequence of 10 balls drawn from the urn with replacement, where \(X_i \in \{\text{red, blue, green}\}\). Let \(R = R(X_1, X_2, \ldots, X_{10})\) be the number of red balls, \(B = B(X_1, X_2, \ldots, X_{10})\) the number of blue balls, and \(G = G(X_1, X_2, \ldots, X_{10})\) the number of green balls in the sequence.

(a) Write down the probability of observing a particular sequence “red, red, red, red, red, blue, blue, blue, green, green.” Does this probability depend on this particular order of the drawing?

(b) Write a general expression for the probability of a sequence with \(R = r\), \(B = b\), and \(G = g\). Evaluate this expression for \(R = 5\), \(B = 3\), and \(G = 2\).

(c) How many sequences of drawings are there with \(R = r\), \(B = b\), and \(G = g\)? Evaluate this number for \(R = 5\), \(B = 3\), and \(G = 2\).

(d) What is the most likely value (are the most likely values) of the sequence \(X_1, \ldots, X_{10}\)? How many sequences are there with this highest probability?

(e) What is the total probability of all sequences with \(R = r\), \(B = b\), and \(G = g\)? Evaluate this total probability for \(R = 5\), \(B = 3\), and \(G = 2\), for \(R = 4\), \(B = 3\), and \(G = 3\) and for \(R = 10\), \(B = 0\), and \(G = 0\).

When 10 balls are drawn with replacement from this urn, what outcome should one “expect?” Explain in one or two sentences.

3. Chapter 2, Problem 13: Suppose one has \(n\) coins, among which there may or may not be one counterfeit coin. If there is a counterfeit coin, it will be either heavier or lighter than the other coins. The coins are to be weighed by a balance.

(a) Find an upper bound on the number of coins \(n\) so that \(k\) weighings will find the counterfeit coin (if any) and correctly declare it to be heavier or lighter.

(b) What is the coin weighing strategy for \(n = 12\) and \(k = 3\) weighings?