ECE 520.447
Introduction to Information Theory and Coding

Midterm Examination #2

9:00 — 10:00 AM, November 20, 2001.

Name: ____

Read these instructions before starting the examination.

(i) This is a open-book examination. Use of any one textbook and your **own** class notes is permitted. Photocopied material from other books, notes of others, *etc.* are not permitted.

(ii) Use of electronic calculators is permitted for numeric calculations only.

(iii) Show all your work clearly and concisely. Points may be deducted for illegible or unclear answers.

(iv) Provide answers in the space provided. Use the unprinted side of the pages in the examination booklet if necessary.

(v) There are three mandatory questions for a total of 50 points.

Best of luck!

Question № 1	/20 Points
Question № 2	/15 Points
Question № 3	/15 Points
TOTAL	/50 Points

Question No 1: Consider a 3 horse race with win probabilities

$$\mathbf{p} = (p_1, p_2, p_3) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Assume that a bookmaker offers o-for-1 odds on this race with $\mathbf{o} = (o_1, o_2, o_3) = (4, 4, 2)$.

(1a) If the true win probabilities are known to a gambler, what is the log-optimal betting strategy $\mathbf{b}^* = (b_1^*, b_2^*, b_3^*)$ for this race, and what is the doubling rate of $\mathbf{b}^*$ (4 points)

(1b) Even if the gambler does not know $\mathbf{p}$, there are several betting strategies for which his or her wealth grows without bounds. Find the *set* of *all* bets $\mathbf{b}$ for which the compounded wealth in repeated races will grow to infinity. You may wish to note that $\sum_{i=1}^{3} \frac{1}{o_i} = 1$ and use the notation $\mathbf{r} = (r_1, r_2, r_3) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$. (6 points)

(1c) If the bookmaker finds out the true win probabilities, what odds $\mathbf{o}' = (o_1', o_2', o_3')$ should be set so as to *minimize* the chances of a gambler if the bookmaker is required by law to have $\sum_{i=1}^{3} \frac{1}{o_i'} \leq 1$? (4 points)

(1d) If a gambler places bets $\mathbf{b}' = (b_1', b_2', b_3') \neq \mathbf{b}^*$ regardless of the odds, find the *set* of *all* odds $\mathbf{o}$ for which the gambler's compounded wealth in repeated races decays to zero, even when the bookmaker required to offer fair odds: $\sum_{i=1}^{3} \frac{1}{o_i} = 1$. (5 points)

Question No 2: Consider a cascade of n mutually independent binary symmetric channels (BSCs), each with crossover probability p.

(2a) If the output of one channel is directly connected to the input of the following channel, show that the resulting channel is equivalent to a single BSC with crossover probability $\frac{1}{2}(1-(1-2p)^n)$. You may use X_0 to denote the input to the first BSC in the cascade, X_1 as the output of the first BSC which also is the input of the second BSC, and so on, up to X_n for the output of the terminal BSC. (Hint: Use induction.) (5 points)

(2b) Show that the capacity of such a cascade of channels goes to zero as n increases. (Hint: What is $\lim_{n\to\infty} I(X_0; X_n)$?) (5 points)

(2c) Note in the setting of 2(a) that only one encoder-decoder pair is used in the system (to map messages to the inputs X_0 and to decode the outputs X_n). If an encoder-decoder pair is used for each BSC, *i.e.*, if BSC_i has inputs X_i and outputs Y_i and a decoder first processes the Y_i's and then selects the X_{i+1} to transmit, what is the capacity of the cascade? (5 points)

Question No 3: The following questions pertain to a binary (15,11) Hamming code.

(3a) Write down a 4×15 parity check matrix for the code in which the first four columns form an identity matrix. (4 points)

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

(3b) Write down the 11×15 generator matrix for the code. (3 points)

$$G = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

(3c) Write down a decoder function $g : \{0,1\}^{15} \to \{0,1\}^{11}$ for this code which will correct all 1-bit errors. (Hint: Use the Hamming distance of the received sequence y^{15} from valid codewords $x^{15}(i)$ to construct g.) (3 points)

(3d) If this code is used over a BSC with crossover probability $p = 0.01$, calculate the probability that a transmitted 11-bit message will be decoded erroneously. (5 points)

Extra Work Space