ECE 520.447
Introduction to Information Theory and Coding

Midterm Examination #1

9:00 — 9:50 AM, October 11, 2001.

Name: ____

Read these instructions before starting the examination.

(i) This is a open-book examination. Use of any one textbook and your **own** class notes is permitted. Photocopied material from other books, notes of others, *etc.* are not permitted.

(ii) Use of electronic calculators is permitted for numeric calculations only.

(iii) Show all your work clearly and concisely. Points may be deducted for illegible or unclear answers.

(iv) Provide answers in the space provided. Use the unprinted side of the pages in the examination booklet if necessary.

(v) There are three mandatory questions for a total of 50 points.

Best of luck!

Question № 1	/10 Points
Question № 2	/20 Points
Question № 3	/20 Points
TOTAL	/50 Points

Question No 1: Consider the 3-state Markov process $U_1, U_2, \ldots$, having tramsition matrix

$\frac{U_n}{U_{n-1}}$	S_1	S_2	S_3
S_1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
S_2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
S_3	0	$\frac{1}{2}$	$\frac{1}{2}$

Thus the probability that S_1 follows S_3 is equal to zero. Note that the steady state distribution for this Markov process is $\mu = \left(\frac{2}{9}, \frac{4}{9}, \frac{3}{9}\right)$.

(1a) Design 3 binary source codes $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ (one each for states S_1, S_2 and S_3), such that this Markov process can be transmitted with maximal compression by the following scheme. (6 points)

1. Note the present symbol S_i.
2. Select code $\mathcal{C}_i$.
3. Note the next symbol S_j and send the codeword in $\mathcal{C}_i$ corresponding to S_j.
4. Repeat for the next symbol.

(1b) What is the average message length of the next symbol conditioned on the previous state $S = S_i$ using this coding scheme? (4 points)

Question No̲ 2: The World Series is a seven game series that terminates as soon as either team wins four games.

- Let X be a random variable that represents the outcome of a World Series between two teams A and B; possible values of X are AAAA, ABABBB, *etc.*
- Let $Y = n(X)$ be the number of games played; *e.g.*, $n(\text{AAAA}) = 4$, $n(\text{ABABBB}) = 6$.
- Let $Z = w(X)$ be the winner of the series; *e.g.*, $w(\text{AAAA}) = \text{A}$, $w(\text{ABABBB}) = \text{B}$.

Assume that the two teams A and B are equally matched and that the outcomes of the games are independent.

(2a) Are the following assertions true or false? Circle your answer. (4 points)

$H(X,Y,Z) = H(X,Y)$	true	false
$H(X,Y,Z) = H(X,Z)$	true	false
$H(X,Y,Z) = H(Y,Z)$	true	false
$H(X,Y,Z) = H(X)$	true	false

(2b) Calculate $H(X|Y = i)$ for $i = 4, 5, 6$ and 7. (4 points)

(2c) Calculate $P(Y = i)$ for $i = 4, 5, 6, 7$, and then calculate $H(Y)$. (4 points)

(2d) Calculate $H(X)$. (4 points)

(2e) Guess the value of $H(X|Z)$. No calculations need be shown! (4 points)

Hint: $\underbrace{H(X, Z)}_{=?} = \underbrace{H(Z)}_{=?} + H(X|Z)$.

Question No̲ 3: Let X_1 and X_2 be discrete random variables drawn according to the probability mass functions $p_1(\cdot)$ and $p_2(\cdot)$ over the respective alphabets $\mathcal{X}_1 = \{1, 2, \ldots, m\}$ and $\mathcal{X}_2 = \{m+1, m+2, \ldots, n\}$. Let

$$X = \begin{cases} X_1 & \text{with probability } \alpha, \\ X_2 & \text{with probability } 1-\alpha. \end{cases}$$

(2a) Find $H(X)$ in terms of $H(X_1)$, $H(X_2)$ and α. (7 points)

(2b) Determine the maximum value of $H(X)$ over α. (6 points)

(2c) Conclude from the result of (2b) that $2^{H(X)} \leq 2^{H(X_1)} + 2^{H(X_2)}$. (2 points)

(2d) Interpret the inequality of (2c) using the notion that $2^{H(X)}$ is the effective alphabet size of a random variable X. (5 points)

Extra Work Space