

ECE 520.447  
Introduction to Information Theory and Coding

Homework #1  
Due in class at 9:00 AM, September 19, 2001.

September 12, 2001

1. Let  $(X, Y)$  be random variables taking values in  $\mathcal{X} \times \mathcal{Y}$ , where  $\mathcal{X}$  and  $\mathcal{Y}$  are discrete finite sets of cardinality  $|\mathcal{X}| = M$  and  $|\mathcal{Y}| = N$ . Let  $P_{XY}$  be the joint p.m.f. of  $(X, Y)$ .
  - (a) What are the maximum possible values of  $H(X, Y)$ ,  $H(X)$ ,  $H(Y|X)$  and  $H(X|Y)$ ?
  - (b) Characterize the p.m.f.  $P_{XY}^*$  which achieves the maximum in each case.
  - (c) Characterize the p.m.f. which yields  $H(X) = H(X|Y)$ .
2. Consider an urn containing 50 red, 30 blue and 20 green balls. Let  $X_1, X_2, \dots, X_{10}$  be a random sequence of 10 balls drawn from the urn *with replacement*, where  $X_i \in \{\text{red, blue, green}\}$ . Let  $R = R(X_1, X_2, \dots, X_{10})$  be the number of red balls,  $B = B(X_1, X_2, \dots, X_{10})$  the number of blue balls, and  $G = G(X_1, X_2, \dots, X_{10})$  the number of green balls in the sequence.
  - (a) Write down the probability of observing a particular sequence “red, red, red, red, red, blue, blue, blue, green, green.” Does this probability depend on this particular order of the drawing?
  - (b) Write a general expression for the probability of a sequence with  $R = r$ ,  $B = b$ , and  $G = g$ . Evaluate this expression for  $R = 5$ ,  $B = 3$ , and  $G = 2$ .
  - (c) How many sequences of drawings are there with  $R = r$ ,  $B = b$ , and  $G = g$ ? Evaluate this number for  $R = 5$ ,  $B = 3$ , and  $G = 2$ .
  - (d) What is the most likely value (are the most likely values) of the sequence  $X_1, \dots, X_{10}$ ? How many sequences are there with this highest probability?
  - (e) What is the *total* probability of *all sequences* with  $R = r$ ,  $B = b$ , and  $G = g$ ? Evaluate this total probability for  $R = 5$ ,  $B = 3$ , and  $G = 2$ , for  $R = 4$ ,  $B = 3$ , and  $G = 3$  and for  $R = 10$ ,  $B = 0$ , and  $G = 0$ .

When 10 balls are drawn with replacement from this urn, what outcome should one “expect?” Explain in one or two sentences.

3. **Chapter 2, Problem 13:** Suppose one has  $n$  coins, among which there may or may not be one counterfeit coin. If there is a counterfeit coin, it will be either heavier or lighter than the other coins. The coins are to be weighed by a balance.
  - (a) Find an upper bound on the number of coins  $n$  so that  $k$  weighings will find the counterfeit coin (if any) and correctly declare it to be heavier or lighter.
  - (b) What is the coin weighing strategy for  $n = 12$  and  $k = 3$  weighings?