ECE 520.651
Random Signals Analysis

Final Examination

9:00 AM — 12:00 PM, December 19, 2003.

Name: ____

Read these instructions before starting the examination.

(i) This is an open-book examination. Use of the Stark and Woods textbook and the Poor textbook (original or photocopy) is permitted. Photocopied material from other books, handwritten/class notes, homework solutions, *etc.* are **not** permitted.

(ii) Use of electronic calculators is permitted for numeric calculations only.

(iii) Show all your work clearly and concisely. Points may be deducted for illegible or unclear answers.

(iv) Write your answers in the space provided. Use the unprinted side of the pages for additional space.

(v) There are four mandatory questions for a total of 80 points.

(vi) If you prefer to receive your final grade via e-mail, *explicitly* provide your e-mail address below; you will receive it via the registrar as usual if you leave this space bank.

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Best of luck!

Question № 1	/20 Points
Question № 2	/20 Points
Question № 3	/20 Points
Question № 4	/20 Points
TOTAL	/80 Points

Question No 1: *Characterization of Random Processes*

(1a) Let $X[n]$ be a real-valued stationary random sequence with mean $E\{X[n]\} = \mu_X$ and autocorrelation function $E\{X[n+m]X[n]\} = R_{XX}[m]$. If $X[n]$ is the input to a D/A converter, the continuous-time output can be idealized as the *analog* random process $X_a(t)$ with

$$X_a(t) = X[n], \qquad \text{for} \qquad n \leq t < n+1, \quad \forall\, t \in \mathbb{R}\,.$$

(i) Find the mean function $E\{X_a(t)\} = \mu_{X_a}(t)$ as a function of μ_X.

(ii) Find the autocorrelation function $E\{X_a(t_1)X_a(t_2)\} = R_{X_a X_a}(t_1, t_2)$ in terms of $R_{XX}[m]$.

(iii) Is the analog signal $X_a(t)$ wide-sense stationary? Elaborate.

Note: To write your answers compactly, you may wish to use the notation $\lfloor t \rfloor$ to denote the largest integer that is not larger than t.

(1b) Let $X(t)$ be a real-valued continuous-time Markov random process, with $t \in [0, \infty)$, initial density $f_X(x; 0)$ at time $t = 0$, and conditional pdf

$$f_X(x_2 | x_1; t_2, t_1) = \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left\{-\frac{1}{2}\frac{(x_2 - x_1)^2}{t_2 - t_1}\right\}, \qquad \forall\ t_2 > t_1 \geq 0.$$

(i) Find $f_X(x; t)$ for an arbitrary time $t > 0$ when $f_X(x; 0) = \delta(x - 1)$.

(ii) Find $f_X(x; t)$ for an arbitrary time $t > 0$ when $f_X(x; 0) = \mathcal{N}(0, 1)$.

Question No 2 *Estimation of Nonrandom Parameters.* Let Y_1 and Y_2 be independent Poisson random variables, each with parameter $\lambda \in \Lambda = (0, \infty)$. Define the parameter θ as

$$\theta = e^{-\lambda}.$$

Consider the problem of estimating θ from $\mathbf{Y} = [\,Y_1 \quad Y_2\,]^T$.

(2a) Show that $T(\mathbf{Y}) = Y_1 + Y_2$ is a complete sufficient statistic for θ.

(2b) Define an estimate $\hat{\theta}$ by

$$\hat{\theta}(\mathbf{Y}) = \frac{1}{2}\left[g(Y_1) + g(Y_2)\right],$$

where $g(\cdot)$ is an indicator function defined as

$$g(y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{if } y \neq 0. \end{cases}$$

Show that $\hat{\theta}$ is an unbiased estimate of θ.

(2c) Find the MVUE of θ.

(2d) Find the maximum-likelihood estimate of θ from $\mathbf{Y}$ and check if it is unbiased.

(2e) Compute the Cramér-Rao bound on the variance of unbiased estimates of θ.

Hint: See Example 5.7-2 on p280 of Stark and Woods for the pmf of $T(\mathbf{Y})$.

Additional Work Space

Question No̲ 3: *Estimation of Random Variables.*

(3a) Use the *orthogonality principle* to show that the minimum mean-squared error

$$\epsilon^2 = E\left[(X - E[X|Y])^2\right]$$

in estimating a real-valued random variable X from a real-valued random variable Y may be expressed as

$$\epsilon^2 = E\left[X\left(X - E[X|Y]\right)\right] ,$$

or as

$$\epsilon^2 = E\left[X^2\right] - E\left[(E[X|Y])^2\right] .$$

Generalize to the case where $\mathbf{X}$ and $\mathbf{Y}$ are real-valued random vectors; *i.e.* show that the MMSE matrix is

$$\begin{aligned} \epsilon^2 &= E\left[(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^T\right] \\ &= E\left[\mathbf{X}(\mathbf{X} - E[\mathbf{X}|\mathbf{Y}])^T\right] \\ &= E\left[\mathbf{X}\mathbf{X}^T\right] - E\left[E[\mathbf{X}|\mathbf{Y}]E^T[\mathbf{X}|\mathbf{Y}]\right] \end{aligned}$$

(3b) Consider the observation model

$$Y_k = \Theta s_k + N_k, \qquad k = 1, 2, \ldots,$$

where $N_1, N_2, \ldots$ are i.i.d. random variables with common density $\mathcal{N}(0, \sigma^2)$, where Θ, with density $\mathcal{N}(\mu, \nu^2)$, is independent of $N_1, N_2, \ldots$, and $s_1, s_2, \ldots$ is a known sequence. Note: Example 9.1-2 on p563 in the Stark and Woods textbook works out the **L**MMSE estimate for a similar problem.

Let $\hat{\theta}_n$ denote the MMSE estimate of Θ given $Y_1, \ldots, Y_n$.

Find recursions for $\hat{\theta}_n$ and for the minimum mean-squared error $E\left[(\Theta - \hat{\theta}_n)^2\right]$ by recasting this as a Kalman filtering problem.

Additional Work Space

Question No 4: *Binary Hypothesis Testing.* Consider a binary hypothesis testing problem in which a $\mathbb{R}$-valued observation Y is generated under the null hypothesis, H_0, according to

$$p_0(y) = \frac{1}{2}e^{-|y|}, \qquad y \in \mathbb{R},$$

and under the alternate hypothesis, H_1, according to

$$p_1(y) = \frac{\theta}{2}e^{-\theta|y|}, \qquad y \in \mathbb{R},$$

for some constant $\theta \geq 2$.

(4a) For $\theta = 2$, find the Bayes decision rule and the minimum Bayes risk for testing H_0 versus H_1 with uniform costs and prior $\pi_0 = \frac{1}{3}$.

(4b) For $\theta = 2$, find the minimax rule and minimax risk for uniform costs.

(4c) For $\theta = 2$, find the Neyman-Pearson rule and the corresponding detection probability for false-alarm probability $\alpha \in (0, 1)$.

(4d) Plot the probability of detection of (4c) as a function of α.

(4e) If θ were fixed but *unknown*, *i.e.*, if we did not assume $\theta = 2$, how would you revise the Neyman-Pearson rule of (4c)? What does this say about the performance of your (revised) rule for testing H_0 versus H_1, when H_1 is a *composite* hypothesis?

Additional Work Space