ECE 520.651 Random Signal Analysis

Homework # 11

Due 9:00 AM on Friday, December 8, 2006.

Review Section 9.1 from Stark and Woods, and Section V.C (p 221-228) from Poor.

1. Fill in the details of the following topics covered in class.

   (a) On the bottom of page 227 in the Poor book, clearly show how

   $$E\left[(X_t - E[X_t])\, Y_l\right] = \sum_{n=a}^{b} h_{t,n} E\left[(Y_n - E[Y_n])\, Y_l\right], \qquad a \le l \le b,$$

   implies

   $$\mathrm{Cov}(X_t, Y_l) = \sum_{n=a}^{b} h_{t,n} \mathrm{Cov}(Y_n, Y_l), \qquad a \le l \le b.$$

   (b) In finding the LMMSE estimate of X based on $Y_a^b = \{Y_a, \ldots, Y_b\}$, make a precise argument why we can assume without loss of generality that X and Y_a^b are zero mean random variables. *Specifically*, if any of $E[X] = \mu_X$ or $E[Y_a^b] = \mu_a^b$ are nonzero, then show how to

      i. construct random variables Z and W_a^b which are zero mean,

      ii. construct the LMMSE estimate $\hat{Z}$ of Z based on W_a^b, and

      iii. (re)construct the LMMSE estimate $\hat{X}$ of X from $\hat{Z}$.

   (c) Do the arguments of 1b carry over to the MMSE (not LMMSE) estimate of X?

2. *Alternate proof of Theorem 9.1-4(a) from Stark and Woods*: Let $\hat{X}_1$ be the LMMSE estimate of X_1 based on $\mathbf{Y} = [Y_a \ldots Y_b]^{\mathrm{T}}$, and $\hat{X}_2$ the LMMSE estimate of X_2 based on $\mathbf{Y}$. Use the *orthogonality principle* to show that the LMMSE estimate $\hat{Z}$ of

   $$Z = X_1 + X_2$$

   based on $\mathbf{Y}$ satisfies:

   $$\hat{Z} = \hat{X}_1 + \hat{X}_2 \qquad \text{almost surely.}$$

   Hint: Compute the difference

   $$E\left[\left(Z - \hat{Z}\right)^2\right] - E\left[\left(Z - (\hat{X}_1 + \hat{X}_2)\right)^2\right]$$

   and use the fact that linear combinations of any two (or all three) of $\hat{X}_1$, $\hat{X}_2$ and $\hat{Z}$ are in $\mathcal{H}_a^b$, the family of affine functions of $\mathbf{Y}$.

3. *Alternate proof of Theorem 9.1-4(b) from Stark and Woods*: Let $\hat{X}_1$ be the LMMSE estimate of X based on $\mathbf{Y}_1 = [Y_a \ldots Y_b]^{\mathrm{T}}$, and $\hat{X}_2$ the LMMSE estimate of X based on $\mathbf{Y}_2 = [Y_c \ldots Y_d]^{\mathrm{T}}$. Use the *orthogonality principle* to show that if $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are *orthogonal* random vectors, then the LMMSE estimate $\hat{X}$ of X based on

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix}$$

satisfies

$$\hat{X} = \hat{X}_1 + \hat{X}_2 \qquad \text{almost surely.}$$

Hint: Assume all random variables are zero mean and, following (V.C.19) on p228 in the Poor book, write the LMMSE estimate of X based on $\mathbf{Y}$ as

$$\begin{aligned} \hat{X} &= \mathbf{h}^{\mathrm{T}}\mathbf{Y} \\ &= \begin{bmatrix} \mathbf{h}_1^{\mathrm{T}} \, \mathbf{h}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} \\ &= \mathbf{h}_1^{\mathrm{T}}\mathbf{Y}_1 + \mathbf{h}_2^{\mathrm{T}}\mathbf{Y}_2, \end{aligned}$$

where $\mathbf{h} = \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{bmatrix}$, $\mathbf{h}_1^{\mathrm{T}}\mathbf{Y}_1 \in \mathcal{H}_a^b$ and $\mathbf{h}_2^{\mathrm{T}}\mathbf{Y}_2 \in \mathcal{H}_c^d$. If $\mathbf{Y}_1 \perp \mathbf{Y}_2$, does it follow that the (scalar) random variables $\mathbf{h}_1^{\mathrm{T}}\mathbf{Y}_1$ and $\mathbf{h}_2^{\mathrm{T}}\mathbf{Y}_2$ are orthogonal for every $\mathbf{h}$?

4. Consider the problem of estimating a random variable X, with $E[X^2] < \infty$, based on the random sequence $\{Y_t\}_{t=0}^{\infty}$. For any fixed n, the MMSE estimate of X based on Y_0^n is a well defined *random variable*:

$$Z_n = E[X|Y_0^n].$$

Show that the sequence $\{Z_n\}_{n=0}^{\infty}$ is a Martingale.

5. Use the *Martingale Convergence Theorem* (Theorem 6.8-4 in Stark and Woods) to prove that the sequence Z_n defined in Problem 4 converges almost surely. Hint: Try to bound $E\left[Z_n^2\right]$ by $E\left[X^2\right]$.

Remark: This (almost sure) limit is defined as the MMSE estimate of X based on the entire sequence $\{Y_t\}_{t=0}^{\infty}$.