Random Signal Analysis 520.651 Fall 2002 Midterm
Department of Electrical and Computer Engineering
The Johns Hopkins University

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NAME: ____

SOCIAL SECURITY NUMBER: ______________________________

INSTRUCTIONS

- Write your name and social security number in the space provided above.
- This exam is closed-book and closed-notes.
- One 8 1/2" by 11" handwritten sheet (both sides) may be used.
- Calculators may be used.
- Give neat step-by-step solutions that include all the details.
- Please put a box around your solution.

SCORING

1	________
2	________
3	________
4	________
Total	________

Problem 1 (30 points)

Let $(X, Y) \sim f_{XY}(x, y)$, where

$$f_{XY}(x,y) = \begin{cases} c & \text{if } x, y \geq 0, x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find c.
(b) Find the marginal pdf's, $f_X(x)$ and $f_Y(y)$.
(c) Find $E[X]$, $\text{Var}[X]$.
(d) Are X and Y uncorrelated? Independent? Explain.
(e) Find $P[X \geq \frac{1}{2}Y]$.
(f) Define another random variable as $Z = |X| + |Y|$. Find the pdf of Z.
(g) Find $E[|X| + |Y|]$.
(h) Find an expression for $E[X|Y = y]$.
(i) Find the pdf for the random variable $W = E[X|Y]$.
(j) Find $E[XY|Y > \frac{1}{2}]$.

Problem 2 (True or False) (20 points)

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ can be a covariance matrix.
(b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ can be a covariance matrix.
(c) In the normal case, the ML estimator is equal to the MVUE estimator.
(d) Two Gaussian r.v.'s that are uncorrelated are also independent.

Problem 3 (20 points)

Consider the estimation of an unknown parameter θ from N samples $x[0]$ through $x[N-1]$:

$$x[n] = \theta r^n + w[n]$$

where r is a known constant with $0 < r < 1$, and $w[n]$ is i.i.d. Gaussian with mean zero and variance σ_w^2.
Hint: $\sum_{n=0}^{N-1} r^{2n} = \frac{1-r^{2N}}{1-r^2}$.

(a) Find the Cramer Rao bound for the variance of an unbiased estimator $\hat{\Theta}$ for θ.
(b) Does an efficient estimator for θ exist? If it does, what is it?
(c) What is the Maximum Likelihood estimator of θ?

Problem 4 (30 points) For each of the following random sequences, determine if the random sequence i) has constant mean; ii)has constant variance; iii) is WSS; iv) has independent increments. **Justify your answers.** In (a) and (b), ω_0 is a known constant.

(a) $X[n] = A\cos(\omega_0 n)$ where A is a Gaussian random variable with mean m_A and variance σ_A^2.
(b) $X[n] = \cos(\omega_0 n + \Phi)$ where Φ is a random variable uniformly distributed over the interval $[0, 2\pi]$.
(c) $X[n] = \sum_{i=0}^{n} W[i]$ where $W[i]$ is an independent Gaussian random sequence with zero mean and unit variance.

Formulas

$$\begin{aligned} \cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\ \cos^2 a &= \frac{1}{2} + \frac{\cos 2a}{2} \\ \cos a \cos b &= \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b) \end{aligned}$$