ECE 520.651 Random Signal Analysis

First (Mid-Term) Examination, Fall 2007

9:00 AM — 12:00 PM, October 25, 2007.

Name: ____

Read these instructions before starting the examination.

(i) This is an open-book examination. Use of any one textbook, Prof Papamarcou's notes and one 8.5×11 sheet of paper ("crib sheet") with formulae written in your own hand is permitted. Photocopied material from additional books, class notes or homework solutions, material obtained via the Internet *etc.* are **not** permitted.

(ii) Use of electronic calculators is permitted for numeric calculations only.

(iii) Show all your work clearly and concisely. Points may be deducted for illegible or unclear answers.

(iv) Write your answers in the space provided. Use the unprinted side of the pages for additional space.

(v) There are five mandatory questions for a total of 100 points. Use the check-list below to keep track of your progress.

Best of luck!

Question № 1 (a) (b)	/20 Points
Question № 2 (a) (b) (c)	/20 Points
Question № 3 (a) (b) (c) (d)	/20 Points
Question № 4 (a) (b)	/20 Points
Question № 5 (a) (b) (c) (d) (e)	/20 Points
TOTAL	/100 Points

Extra work-space 1

Question No̲ 1: *Sample Spaces, Event Spaces and Independent Events*: Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(1a) Let $\mathcal{G}$ be a collection of subsets of an event $B \in \mathcal{F}$ given by

$$\mathcal{G} = \left\{ \tilde{A} \, : \, \tilde{A} = A \cap B \text{ for some } A \in \mathcal{F} \right\}.$$

Show that $\mathcal{G}$ constitutes an event space on the sample space B. (10 points)

(1b) For some prime number p, let $\Omega = \{1, 2, \ldots, p\}$, $\mathcal{F} = 2^{\Omega}$ and let P be the uniform probability assignment on Ω. Let X and Y be $\mathbb{R}$-valued random variables on Ω. If X is independent of Y, show that at least one of them must be a constant. (10 points)

Extra work-space 2

Question N$\underline{\text{o}}$ 2 *Conditional probability given multiple events*: Given a probability space $(\Omega, \mathcal{F}, P)$, let $B_1, B_2 \in \mathcal{F}$ with $P(B_1) > 0$, $P(B_2) > 0$ and $P(B_1 \cap B_2) > 0$.

(2a) If $Q_1 : \mathcal{F} \to [0, 1]$ is defined as

$$Q_1(A) = \frac{P(A \cap B_1)}{P(B_1)} \qquad \forall A \in \mathcal{F},$$

Show that $(\Omega, \mathcal{F}, Q_1)$ is a probability space. i.e. show that Q_1 satisfies axioms **P1–P3** on page 18 of Prof. Papamarcou's notes. (12 points)

(2b) If $Q_{12} : \mathcal{F} \to [0, 1]$ and $Q_{21} : \mathcal{F} \to [0, 1]$ are defined as (6 points)

$$Q_{12}(A) = \frac{Q_1(A \cap B_2)}{Q_1(B_2)} \quad \text{and} \quad Q_{21}(A) = \frac{Q_2(A \cap B_1)}{Q_2(B_1)}, \quad \text{where} \quad Q_2(A) = \frac{P(A \cap B_2)}{P(B_2)},$$

show that $Q_{12}(A) = Q_{21}(A)$ for all events $A \in \mathcal{F}$.

(2c) Interpret Q_{12} (or Q_{21}) as a conditional probability derived from P. (2 points)

Conclude that for computing conditional probabilities, if we first condition on an event B_1 and then on an event B_2, or *vice versa*, we end up with the same probability assignment.

Extra work-space 3

Question No 3: *The Borel-Cantelli Lemmas and Convergence of Random Variables*: Let A_n, $n = 1, 2, \ldots$, be a sequence of events in some probability space $(\Omega, \mathcal{F}, P)$, and let

$$A = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

represent the event that infinitely many of the A_n occur. Show that

(3a) if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A) = 0$; (4 points)

(3b) if $\sum_{n=1}^{\infty} P(A_n) = \infty$ and the A_n are mutually independent, then $P(A) = 1$; (6 points)

(3c) the statement above is false if the independence assumption is dropped; (3 points)

(3d) if $\sum_{n=1}^{\infty} E\left[|X_n - X|^r\right] < \infty$ for some $r > 0$, then $X_n \to X$ almost surely. (7 points)

Therefore, if X_n converges in the r-th mean *sufficiently fast*, then it converges almost surely.

Hints: $P(E \cup F) \leq P(E) + P(F)$ and more generally $P(\cup_n E_n) \leq \sum_n P(E_n)$, while for independent events, $P(E \cap F) = P(E)P(F)$ and more generally $P(\cap_n E_n) = \prod_n P(E_n)$.

Extra work-space 4

Extra work-space 5

Question N$\underline{\text{o}}$ 4: *Computing Probability Densities of Functions of Random Variables*: Let X and Y be independent and identically, exponentially distributed random variables with common rate parameter λ.

(4a) Find the joint density of $Z = X + Y$ and $W = \frac{X}{Y}$, and deduce that Z and W are independent. (10 points)

(4b) For $\lambda = 1$, find the joint density of $U = X + Y$ and $V = \frac{X}{X+Y}$. What is the marginal density of V? (10 points)

Extra work-space 6

Question N$\underline{o}$ 5: *Moment Generating Functions and Random Walks*: Starting at the origin at time $n = 0$, let

$$X_n = X_{n-1} + W_n, \quad n = 1, 2, \dots,$$

represent an integer-valued random walk, where the W_n are independent and identically distributed random variables with common pmf

$$P(W = +1) \;=\; p \qquad \text{and} \qquad P(W = -1) \;=\; q, \qquad \text{where } q = 1 - p.$$

Let $T > 0$ denote the epoch at which the random walk returns to the origin for *the first time*, and calculate $P(T = n)$ using moment generating functions, as suggested below.

(5a) Let $R(n) = P(X_n = 0)$ be the probability that the random walk visits the origin—not necessarily for the first time—at time n. Explain why

$$\forall\, n > k > 0, \qquad P(X_n = 0 | T = k) = R(n - k).$$

Use it to show that $R(n) = \sum_{k=1}^{n} R(n-k)P(T = k)$ for $n = 1, 2, \dots$. (4 points)

(5b) Show that $M_R(s) = 1 + M_T(s)M_R(s)$, where $M_R(s) = \sum_{n=0}^{\infty} R(n)e^{sn}$ and $M_T(s) = E\left[e^{sT}\right]$ is the moment generating function of T. (5 points)

(5c) Show that (5 points)

$$R(n) = \begin{cases} \binom{n}{\frac{n}{2}} (pq)^{\frac{n}{2}} & \text{for even } n, \text{ and} \\ 0 & \text{for odd } n. \end{cases}$$

(5d) Show that $M_R(s) = \left(1 - 4pqe^{2s}\right)^{-\frac{1}{2}}$. (4 points)

(5e) Deduce from the above that $M_T(s) = 1 - \sqrt{1 - 4pqe^{2s}}$. (2 points)

The desired $P(T = n)$ may be computed by inverting this moment generating function.

Extra work-space 7

Extra work-space 8