ECE 520.651 Random Signal Analysis

Second (Final) Examination, Fall 2007

9:00 AM — 12:00 PM, December 20, 2007.

Name: ____

Read these instructions before starting the examination.

(i) This is an open-book examination. Use of any two textbook, Prof Papamarcou's notes, homework solutions *provided in this class* and class-notes *written in your own hand* is permitted. Photocopied material from additional books, notes or solutions to problems written by others or other material obtained via the Internet *etc.* are **not** permitted.

(ii) Use of electronic calculators is permitted for numeric calculations only.

(iii) Show all your work clearly. Points may be deducted for illegible or unclear answers.

(iv) Write your answers in the space provided. Use the unprinted side of the pages if needed.

(v) There are five mandatory questions for a total of 100 points. Use the check-list below to keep track of your progress.

Best of luck!

Question No 1: (a) (b) (c) (d) (e) (f)	/20 Points
Question No 2: (a) (b) (c) (d) (e)	/20 Points
Question No 3: (a) (b) (c)	/20 Points
Question No 4: (a) (b)	/20 Points
Question No 5: (a) (b) (c) (d)	/20 Points
TOTAL	/100 Points

Question No̲ 1: *Markov Chains.* A die with six sides is rolled repeatedly. Which of the following are Markov chains? For those that aren't, explain why, and for those that are, specify the *state space* and the *transition probability matrix*.

(a) The largest number M_n seen up to the n-th roll.

(b) The number N_n of sixes in n rolls.

(c) At time n, the time C_n since the most recent six.

(d) At time n, the time B_n until the next six.

Let X_n be a Markov chain taking values in $\mathcal{S}$ and $h : \mathcal{S} \to \mathcal{T}$ be a one-to-one mapping.

(e) Show that $Y_n = h(X_n)$ is a Markov chain taking values in $\mathcal{T}$.

(f) Does this statement hold if h is not one-to-one?

Extra work-space 1

Question No 2 *Compound Poisson Processes.* Let $N(t)$, $t \geq 0$, be a Poisson process with constant rate λ, and let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables with common density $f_Y(y)$ and characteristic function $\Phi_Y(\omega)$. Let the Y_n's also be independent of $N(t)$. The process

$$\forall\, t \geq 0 \qquad N^*(t) \;=\; Y_1 + Y_2 + \cdots + Y_{N(t)} \;=\; \sum_{n=1}^{N(t)} Y_n$$

is called a *compound* Poisson process; when $N(t) = 0$, the sum is vacuous and $N^*(t) = 0$. This question concerns computation of the probability distribution function of $N^*(t)$.

(a) Compute the characteristic function $\Phi_N(\omega\,;\,t)$ of $N(t)$.

(b) Use this to compute the characteristic function $\Phi_{N^*}(\omega\,;\,t)$ of $N^*(t)$ in terms of $\Phi_Y(\cdot)$. (Hint: try using conditional expectations.)

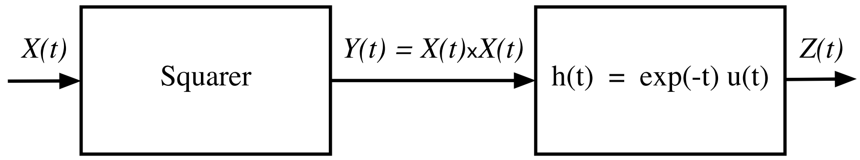
(c) Use (b) to compute the probability distribution of $N^*(t)$ for the case of $f_Y(y) = \delta(y-1)$. Explain why your answer is as expected!

(d) Compute the probability distribution of $N^*(t)$ when $Y \sim$ Bernoulli(p).

(e) Use (d) to argue that if each "arrival" epoch of a Poisson point process is randomly retained (or discarded), the retained epochs continue to form a Poisson point process; a phenomenon often called Poisson *thinning*. What is the rate of the thinned process?

Extra work-space 2

Question No 3: *LTI and non-LTI Systems with Stochastic Inputs.* Let $X(t)$, the input to the system shown in the figure below, be a stationary, zero-mean Gaussian random process.



The power spectral density of $Z(t)$ is measured experimentally and found to be

$$S_{ZZ}(\omega) \quad = \quad \pi\delta(\omega) + \frac{2\beta}{(\omega^2 + \beta^2)(\omega^2 + 1)}.$$

This question concerns recovering the probability distribution of the input process $X(t)$.

(a) Compute the autocorrelation function of $Y(t)$ in terms of β.

(b) Compute the autocorrelation function of $X(t)$. (Hint: see Problem 4.44 in S & W.)

(c) For any $k \geq 1$ and $t_1 \leq t_2 \leq \ldots \leq t_k$, write the joint density of $X(t_1), X(t_2), \ldots, X(t_k)$.

Extra work-space 3

Question No 4: *Estimation of (Fixed) Parameters from Data.* Suppose that we observe[1]

$$Y_k \;=\; A\sin\left(\frac{k\pi}{2}+\Phi\right)+N_k\,, \qquad k=1,2,\ldots,4n,$$

where $N_1, N_2, \ldots, N_{4n}$ are i.i.d. Gaussian r.v.'s with zero mean and known variance σ^2.

(a) Assume A and Φ to be fixed but unknown parameters, $A \geq 0$ and $\Phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and compute their maximum likelihood estimates.

(b) Assume A and Φ to be independent random variables, with (prior) probabilities

$$w_\Phi(\phi) = \begin{cases} \frac{1}{\pi} & \text{if } -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad w_A(a) = \begin{cases} \frac{a}{\beta^2} e^{-\frac{a^2}{2\beta^2}} & \text{if } a \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where β is a known *hyper-parameter*, and compute their MAP estimates.

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[1]Computations simplify if the number of observations is even, and notation simplifies if it is a multiple of 4.

Extra work-space 4

Question N$\underline{\text{o}}$ 5: *Signal Estimation and Kalman Filtering.* Consider the observation model

$$Y_k \;=\; \Theta\, s_k + N_k\,, \qquad k = 1, 2, \ldots,$$

where $N_1, N_2, \ldots$ are i.i.d. $\mathcal{N}(0, \sigma^2)$ r.v.'s, $\Theta \sim \mathcal{N}(\mu, \nu^2)$ is independent of $N_1, N_2, \ldots$, and $s_1, s_2, \ldots$ is a known carrier signal.

(a) Find a recursive formula for $\widehat{\theta}_n$, the MMSE estimate of Θ from $Y_1, Y_2, \ldots, Y_n$, by casting this as a Kalman filtering problem.

(b) Find a recursive formula for the minimum mean-squared error $E\left[(\widehat{\theta}_n - \Theta)^2\right]$.

(c) Draw a schematic (block) diagram for implementing your answers to part (a) and (b).

(d) Qualitatively interpret your answer to (a) and (b).

Extra work-space 5