ECE 520.651 Random Signal Analysis

First (Mid-Term) Examination, Fall 2005

6:00 PM — 8:30 PM, October 27, 2005.

Name: ____

Read these instructions before starting the examination.

(i) This is an open-book, open-notes examination. Use of any one textbook, Prof Papamarcou's notes, and any/all of your own hand-written notes is permitted. Photocopied material from additional books, class notes or homework solutions prepared by others, material obtained via the Internet *etc.* are **not** permitted.

(ii) Use of electronic calculators is permitted for numeric calculations only.

(iii) Show all your work clearly and concisely. Points may be deducted for illegible or unclear answers.

(iv) Write your answers in the space provided. Use the unprinted side of the pages for additional space.

(v) There are five mandatory questions for a total of 100 points. Use the check-list below to keep track of your progress.

Best of luck!

Question № 1 (a) (b) (c) (d)	/20 Points
Question № 2 (a) (b) (c) (d)	/20 Points
Question № 3 (a) (b) (c) (d)	/20 Points
Question № 4 (a) (b)	/20 Points
Question № 5 (a) (b) (c)	/20 Points
TOTAL	/100 Points

Extra work-space 1

Question No̲ 1: *σ-Fields, Probabilities and Independence.* Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, and consider the collection $\mathcal{G}_1 = \{A, B\}$ of subsets of Ω, where $A = \{1, 2, 4\}$ and $B = \{2, 3, 5\}$.

(1a) Construct $\mathcal{F}_1 = \sigma(\mathcal{G}_1)$, the smallest σ-field containing $\mathcal{G}_1$. (5 points)

(1b) Specify a probability assignment $P_1 : \mathcal{F}_1 \to [0, 1]$ such that A and B are *independent* events in the probability space $(\Omega, \mathcal{F}_1, P_1)$. (5 points)

Let $\mathcal{G}_2 = \{A, B, C\}$, where A and B are as specified earlier, and $C = \{1, 5\}$.

(1c) Construct $\mathcal{F}_2 = \sigma(\mathcal{G}_2)$, the smallest σ-field containing $\mathcal{G}_2$. (5 points)

(1d) *Extend* the probability assignment P_1 to $\mathcal{F}_2$ in a consistent manner. i.e specify a probability assignment $P_2 : \mathcal{F}_2 \rightarrow [0, 1]$ such that for any set $D \in \mathcal{F}_1$, $P_2(D) = P_1(D)$. In particular, A and B should still be independent events in $(\Omega, \mathcal{F}_2, P_2)$. (5 points)

Question N$\underline{\text{o}}$ 2 *Conditional Expectation.* Consider a pair of random variables (X, Y) in $\mathbb{R} \times \mathbb{R}$, with the joint pdf

$$f_{X,Y}(x, y) = e^{-2|y-x^2| - x} u(x),$$

where $u(\cdot)$ is the unit step function.

(2a) Compute the marginal density $f_X(x)$ of the random variable of X. Be careful in specifying it for all $x \in \mathbb{R}$. (5 points)

(2b) Compute the conditional density $f_{Y|X}(y|x)$ of Y given $X = x$. Be careful in specifying it for all $y \in \mathbb{R}$. Is it well defined for all $x \in \mathbb{R}$? (5 points)

(2c) Compute the conditional expectation $E[Y|X]$. (5 points)

(2d) Compute the expectation $E[Y]$. (5 points)

Extra work-space 2

Question No̲ 3: *Characteristic Functions of Sums of Random Variables.* Consider the weighted sum

$$S \;=\; \sum_{k=1}^{n} a_k X_k \;=\; \mathbf{a}^{\mathrm{T}}\mathbf{X}.$$

This problem is concerned with the characteristic function $\Phi_S(\omega)$ of S.

(3a) For a general random vector $\mathbf{X} = [X_1, X_2, \ldots, X_n]^{\mathrm{T}}$, show that (3 points)

$$\Phi_S(\omega) \;=\; \Phi_{\mathbf{X}}(\omega\mathbf{a}) \;=\; \Phi_{X_1,\ldots,X_n}\left([a_1\omega, a_2\omega, \ldots, a_n\omega]^{\mathrm{T}}\right), \qquad \forall\,\omega \in \mathbb{R}\,.$$

(3b) If the random variables $X_1, X_2, \ldots, X_n$ are independent, show that (4 points)

$$\Phi_S(\omega) \;=\; \prod_{k=1}^{n} \Phi_{X_k}(a_k\omega)\,, \qquad \forall\,\omega \in \mathbb{R}\,.$$

(3c) If $X_1, X_2, \ldots, X_n$ are i.i.d., and S has constant weights $a_k = a$, show that (4 points)

$$\Phi_S(\omega) \;=\; \Phi_{X_1}^n(a\omega), \qquad \forall\, \omega \in \mathbb{R}.$$

(3d) Compute $\Phi_S(\omega)$ in (3a), (3b) and (3c) when $\mathbf{X} \sim \mathcal{N}(\mathbf{m}, \mathbf{K})$. (9 points)

Extra work-space 3

Question N$\underline{\text{o}}$ 4: *Independent Components Analysis.* Show that every Gaussian random vector may be written as an *affine* transform of a Gaussian random vector with *independent* components, each with zero mean and unit variance. In other words, show that for every n-dimensional Gaussian random vector $\mathbf{X} \sim \mathcal{N}(\mathbf{m}, \mathbf{K})$, there exists a dimension $k \leq n$, a matrix $\mathbf{A}_{n \times k}$, a vector $\mathbf{b}_{n \times 1}$ and a k-dimensional random vector $\mathbf{Y} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ such that

$$\mathbf{X} = \mathbf{AY} + \mathbf{b}.$$

(4a) Show this by explicitly constructing $(\mathbf{A}, \mathbf{b})$ when $\mathbf{K}$ is nonsingular. (12 points)

(4b) Extend the result of (4a) to the case when **K** has rank $k < n$. (8 points)

Question No̲ 5: *Convergence of the Empirical Distribution.* Given i.i.d. observations $X_1, X_2, \ldots$, and some $a \in \mathbb{R}$, let

$$Y_n \;=\; \hat{F}_n(a) \;=\; \frac{1}{n}\sum_{k=1}^{n} u(a - X_k), \qquad n = 1, 2, \ldots,$$

where $u(\cdot)$ is the unit-step function. Let the common CDF of the X_n's be $F(\cdot)$

1. If we fix an n and a particular *realization* $x_1, x_2, \ldots, x_n$, of the random sequence, then $\hat{F}_n(a)$ viewed as a function of a is a valid CDF. (Convince yourself about this fact.)

2. If we fix an a, and view $\hat{F}_n(a)$ as a $\mathbb{R}$-valued function of the r.v's $X_1, X_2, \ldots, X_n$, then $\{Y_n,\, n = 1, 2, \ldots\}$ is a $\mathbb{R}$-valued *random sequence.*

3. If we view $\hat{F}_n$ as a "CDF"-valued function of $X_1, X_2, \ldots, X_n$, then $\{\hat{F}_n,\, n = 1, 2, \ldots\}$ is a *sequence of random CDFs.*

Let us take the second of these three views, i.e. fix an a and think of the random sequence Y_n, $n = 1, 2, \ldots$, for the following questions.

(5a) Does $\hat{F}_n(a)$ converge almost surely? For every value of a? (8 points)

(5b) Does $\hat{F}_n(a)$ converge in r-th mean? For every value of a? (8 points)

(5c) Does $\hat{F}_n(a)$ converge pointwise, in probability or in distribution? (4 points)

Extra work-space 4