ECE 520.651 Random Signal Analysis

Second (Final) Examination, Fall 2005

9:00 AM — 12:00 PM, December 22, 2005.

Name: ____

Read these instructions before starting the examination.

(i) This is an open-book examination. Use of the Stark and Woods textbook, the Poor textbook and Prof. Papamarcou's notes, as well as notes *written by your own hand* are permitted. Additional photocopied material from other books, notes or solutions prepared by others, material obtained via the Internet, *etc.* are **not** permitted.

(ii) You will not need an electronic calculator for this exam!

(iii) Show all your work clearly and concisely. Points may be deducted for illegible or unclear answers.

(iv) Write your answers in the space provided. Use the unprinted side of the pages for additional space.

(v) There are five mandatory questions for a total of 100 points and a bonus question for 20 points. Use the check-list below to keep track of your progress.

Best of luck!

Question № 1 (a) (b) (c)		/20 Points
Question № 2 (a) (b) (c) (d)		/20 Points
Question № 3 (a) (b) (c)		/20 Points
Question № 4 (a) (b)		/20 Points
Question № 5 (a) (b) (c) (d) (e)		/20 Points
Question № 6 (a) (b)	Bonus	/20 Points
TOTAL		/100 Points

Question No̲ 1: *Markov Chains.* Let X_n, $n = 0, 1, 2, \ldots$, be a random sequence taking values in a discrete and finite set $\mathcal{S} = \{1, 2, \ldots, K\}$.

(1a) Show that the following two definitions of a Markov chain are equivalent. (8 points)

1. For all $n \geq 0$ and $x_0, x_1, \ldots, x_n, x_{n+1} \in \mathcal{S}$, it holds that

$$P\left(X_{n+1} = x_{n+1} \,|\, X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n\right) = P\left(X_{n+1} = x_{n+1} \,|\, X_n = x_n\right).$$

2. For all $m, n \geq 0$ and $x_0, x_1, \ldots, x_n, x_{n+m} \in \mathcal{S}$, it holds that

$$P\left(X_{n+m} = x_{n+m} \,|\, X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n\right) = P\left(X_{n+m} = x_{n+m} \,|\, X_n = x_n\right).$$

In other words, show that each of the definitions above implies the other.

(1b) If X_n, $n = 1, 2, \ldots$, is a sequence of independent and identically distributed random variables, which of the following sequences constitute Markov chains? (8 points)

1. the running maximum $M_n = \max\{X_1, X_2, \ldots, X_n\}$, $n = 1, 2, \ldots$
2. the sample sum $S_n = \sum_{k=1}^{n} X_k$, $n = 1, 2, \ldots$
3. the output of a simple FIR filter: $Y_n = X_n + X_{n-1}$, with $X_0 = 0$, $n = 1, 2, \ldots$
4. the number of times a symbol s is seen: $C_n = \sum_{k=1}^{n} \mathbf{1}(X_k = s)$, $n = 1, 2, \ldots$

(1c) Show that for a Markov chain X_n, $n = 0, 1, 2, \ldots$, (4 points)

$$\begin{aligned} & P\left(X_r = x_r \,|\, X_0 = x_0, \ldots, X_{r-1} = x_{r-1}, X_{r+1} = x_{r+1}, \ldots, X_n = x_n\right) \\ & \qquad = \; P\left(X_r = x_r \,|\, X_{r-1} = x_{r-1}, X_{r+1} = x_{r+1}\right), \end{aligned}$$

for all $0 < r < n$ and $x_0, \ldots, x_n \in \mathcal{S}$.

Question No̲ 2: *Martingales.* Recall that a random sequence S_n, $n = 0, 1, 2, \ldots$, is said to be a Martingale if

$$E\left[|S_n|\right] < \infty \quad \text{and} \quad E\left[S_{n+1} \,|\, S_0, S_1, \ldots, S_n\right] = S_n, \qquad \forall\, n \geq 0.$$

(2a) Show that for a Martingale S_n, $n = 0, 1, 2, \ldots$, (5 points)

$$E\left[S_{n+m} \,|\, S_0, S_1, \ldots, S_n\right] = S_n, \qquad \forall\, m, n \geq 0.$$

(2b) Show that Martingales have a constant mean, i.e. $E\left[S_n\right] = E\left[S_0\right]$, $\forall\, n \geq 0$. (5 points)

(2c) If X_n, $n = 0, 1, 2, \ldots$, is a sequence of zero-mean i.i.d. random variables, show that

$$S_n = \sum_{k=1}^{n} X_k, \qquad n \geq 0,$$

is a Martingale. (5 points)

(2d) If X_n, $n = 0, 1, 2, \ldots$, is a time-homogeneous Markov chain taking values in a countable set $\mathcal{S}$, and $\psi : \mathcal{S} \to \mathbb{R}$ is a bounded function which satisfies

$$\sum_{j \in \mathcal{S}} p_{ij} \psi(j) = \psi(i), \qquad \forall\, i, j \in \mathcal{S},$$

where the $p_{ij} = P(X_{n+1} = j \,|\, X_n = i)$ denote transition probabilities of the Markov chain, show that $S_n = \psi(X_n)$ is a Martingale with respect to X_n. (5 points)

Note: A random sequence S_n, $n = 0, 1, 2, \ldots$, is said to be a Martingale with respect to another random sequence X_n if

$$E\left[|S_n|\right] < \infty \quad \text{and} \quad E\left[S_{n+1} \,|\, X_0, X_1, \ldots, X_n\right] = S_n, \qquad \forall\, n \geq 0.$$

Question No̲ 3: *Stationary Random Sequences.* Let X and Y be two uncorrelated random variables, each with zero mean and unit variance, and define the random sequence

$$Z[n] = X \cos \omega n \, + \, Y \sin \omega n, \qquad n = 0, 1, 2, \ldots$$

for some arbitrary parameter $\omega \in [0, \pi]$.

(3a) Show that $Z[n]$ is a *wide sense stationary* sequence. (8 points)

(3b) Show that $Z[n]$ is not *strictly stationary* in general. (4 points)

(3c) If X and Y are jointly Gaussian, show that $Z[n]$ is *strictly stationary*. (8 points)

Question No 4: *Kalman Filtering*. Following the notation in Poor p209-212, let

$$\begin{aligned}
\hat{X}_{t|t} &= E\left[X_t | Y_0, \ldots, Y_t\right] = \hat{X}_{t|t-1} + K_t(Y_t - H_t \hat{X}_{t|t-1}) \\
\hat{X}_{t+1|t} &= E\left[X_{t+1} | Y_0, \ldots, Y_t\right] = F_t \hat{X}_{t|t} \\
\Sigma_{t|t} &= E\left[\left(X_t - \hat{X}_{t|t}\right)^2\right] = \Sigma_{t|t-1} - K_t H_t \Sigma_{t|t-1} \\
\Sigma_{t+1|t} &= E\left[\left(X_{t+1} - \hat{X}_{t+1|t}\right)^2\right] = F_t \Sigma_{t|t} F_t^{\mathrm{T}} + G_t Q_t G_t^{\mathrm{T}}
\end{aligned}$$

(4a) Carefully derive the update equation for the error covariance $\Sigma_{t|t}$, namely (10 points)

$$\Sigma_{t|t} = \Sigma_{t|t-1} - K_t H_t \Sigma_{t|t-1}.$$

(4b) Suppose that the state equation V.B.13a in the Kalman-Bucy model is modified as

$$X_{n+1} = F_n X_n + G_n U_n + \Gamma_n s_n, \qquad n = 0, 1, 2, \ldots,$$

where the matrices Γ_n are known, and the sequence s_n is a known deterministic sequence, e.g. a control signal applied to the linear stochastic system. Derive the modified Kalman filtering equations corresponding to V.B.14, V.B.15 and V.B.16. (10 points)

Question No̲ 5: *Minimum Squared Error (MSE) Estimates.* Let $\mu \in (0, \infty)$ be a fixed but unknown parameter in a parametric family of pdf's

$$f_\mu(y) = \frac{1}{\mu} \exp\left\{-\frac{y}{\mu}\right\} u(y), \qquad y \in \mathbb{R},$$

and let $Y_1^n \equiv Y_1, \ldots, Y_n$, be i.i.d. with common distribution $f_\mu(\cdot)$. Let us consider the problem of finding estimators that minimize the squared error in estimating μ from Y_1^n.

(5a) Compute the Cramér-Rao lower bound for the variance $E_\mu\left[(\hat{\mu}(Y_1^n) - \mu)^2\right]$ of an unbiased estimator $\hat{\mu}(Y_1^n)$. (4 points)

(5b) Compute the maximum likelihood estimate $\hat{\mu}_{\mathrm{ML}}(Y_1^n)$. (4 points)

(5c) Show that $\hat{\mu}_{\mathrm{ML}}(Y_1^n)$ is the MVUE by computing its bias and variance. (4 points)

(5d) Compute the mean squared error of $\tilde{\mu}(Y_1^n) = \frac{1}{n+1} \sum_{k=1}^{n} Y_k$. (4 points)

(5e) Discuss the bias vs variance trade-off problem via $\hat{\mu}_{\mathrm{ML}}(Y_1^n)$ and $\tilde{\mu}(Y_1^n)$. (4 points)

Question No̲ 6 *Simple Binary Hypothesis Testing.*

(6a) Given *one* observation $Y = N + \theta\lambda$, where N is uniformly distributed in $[-1, +1]$, θ is either 0 or 1, and $\lambda \in (0, 2)$ is a fixed (and known) parameter, design a level-α Neyman-Pearson test for deciding between

$$\begin{aligned} H_0 &: \theta = 0 \\ H_1 &: \theta = 1 \end{aligned}$$

for $\alpha \in [0, 1]$, and write the power of the test as a function of α and λ. (10 points)

(6b) Given costs $C_{00} = C_{11} = 0$, $C_{10} = 1$ and $C_{01} = N$, we wish to design the Bayes test for a uniform prior, and the minimax test, for deciding between

$$\begin{aligned} H_0 &: \quad p_0(y) = \tfrac{1}{\sqrt{2\pi}} \exp\left\{-\tfrac{y^2}{2}\right\} \\ H_1 &: \quad p_1(y) = \tfrac{1}{\sqrt{2\pi}} \exp\left\{-\tfrac{(y-1)^2}{2}\right\}. \end{aligned}$$

Discuss the test, and the Bayes or minimax risk, when N is very large. (10 points)