ECE 520.651 Random Signal Analysis

First (Mid-Term) Examination, Fall 2004

6:00 PM — 8:30 PM, October 20, 2004.

Name: ____

Read these instructions before starting the examination.

(i) This is an open-book, open-notes examination. Use of any one textbook, Prof Papamarcou's notes, homework solutions provided in this course, and any/all of your own hand-written notes is permitted. Photocopied material from additional books, class notes or homework solutions prepared by others, material obtained via the Internet *etc.* are **not** permitted.

(ii) Use of electronic calculators is permitted for numeric calculations only.

(iii) Show all your work clearly and concisely. Points may be deducted for illegible or unclear answers.

(iv) Write your answers in the space provided. Use the unprinted side of the pages for additional space.

(v) There are five mandatory questions for a total of 100 points. Use the check-list below to keep track of your progress.

Best of luck!

Question № 1 (a) (b) (c)	/20 Points
Question № 2 (a) (b) (c) (d) (e)	/20 Points
Question № 3 (a) (b)	/20 Points
Question № 4 (a) (b)	/20 Points
Question № 5 (a) (b) (c)	/20 Points
TOTAL	/100 Points

Extra work-space 1

Question N$\underline{o}$ 1: *Probability, Conditional Probability and Sampling from a Population.* Consider all families with exactly two children each. If we use b to denote a boy and g a girl, then the genders of the two childern in a family are one of $\{bb, bg, gb, gg\}$, where the first letter denotes the gender of the older child, and the second letter the gender of the younger one. Assume that the four possibilities are equally likely for every family.

(1a) Given that a family sampled from this population has a *girl*, calculate the conditional probability that her sibling is also a *girl*. (7 points)

(1b) Given that a child sampled from this population is a *girl*, calculate the conditional probability that her sibling is also a *girl*. (7 points)

(1c) Discuss why the two probabilities should or should not be the same, as suggested by your answers above. (6 points)

Question No 2 *Polya's Contagion Models.* An urn contains b black and r red balls. A ball is drawn at random. It is replaced and, moreover, c balls of the color drawn are added to the urn. A new random drawing is made from the urn, now containing $b + r + c$ balls, and this procedure is repeated.

(2a) What is the conditional probability of a black ball at the second drawing, given that the ball at the first drawing was black? (4 points)

(2b) What is the marginal probability of a black ball at the second drawing? (4 points)

(2c) What is the conditional probability of a black ball at the first drawing, given that the ball at the second drawing was black? (4 points)

(2d) Argue why any sequence of n drawings resulting in n_1 black and n_2 red balls has the same probability as the event of first drawing n_1 consecutive black balls and then n_2 consecutive red balls: (4 points)

$$\frac{b}{(b+r)}\frac{(b+c)}{(b+r+c)}\frac{\dots}{\dots}\frac{(b+n_1c-c)}{(b+r+n_1c-c)}\,\frac{r}{(b+r+n_1c)}\frac{(r+c)}{(b+r+n_1c+c)}\frac{\dots}{\dots}\frac{(r+n_2c-c)}{(b+r+nc-c)}.$$

(2e) Show by induction that the marginal probability of a black ball at *any* drawing is $\frac{b}{b+r}$ (4 points)

Remark: The case $c = 0$ is the usual case of sampling with replacement, while $c = -1$ yields sampling without replacement. The cases $c > 0$ model the effects of a contagion, e.g. in communicable diseases, where an outcome temporarily increases the chances of its recurrence. Ploya's urn models have many interesting applications, including stochastic models of neuronal physiology.

Extra work-space 2

Question No̲ 3: *Convergence of Random Variables.* We have shown in class that

- $X_n \to X$ in probability $\quad\Longrightarrow\quad X_n \to X$ in distribution, and
- $X_n \to X$ in r-th mean $\quad\Longrightarrow\quad X_n \to X$ in probability,

but the converse is not *generally* true in either case. This problem investigates *special cases* in which the converse *is* true.

(3a) Show that if $X_n \to c$ in distribution, where c is a constant, then $X_n \to c$ in probability. In other words, if (10 points)

$$\lim_{n\to\infty} F_{X_n}(x) = u(x-c) \qquad \forall\, x \neq c,$$

where $u(\cdot)$ is the unit step function, then show that

$$\lim_{n\to\infty} P\left(|X_n - c| > \epsilon\right) = 0 \qquad \forall\, \epsilon > 0.$$

(3b) Show that if $X_n \to X$ in probability and the X_n's are uniformly bounded, then $X_n \to X$ in r-th mean. In other words, for some constant $c < \infty$, if (10 points)

$$P(|X_n| \leq c) = 1 \quad \forall\, n,$$

then show that

$$\lim_{n\to\infty} E\left(|X_n - X|^r\right) = 0 \qquad \forall\, \epsilon > 0.$$

Question No 4: *Linear Transformations of Gaussian Random Vectors.* Theorem 5-6.2 in the Stark and Woods book states that if $\mathbf{X}$ is an n-dimensional Normal random vector with positive definite covariance matric $\mathbf{K}$ and mean vector $\boldsymbol{\mu}$, and, for some $m \leq n$, $\mathbf{A}$ is an $m \times n$ matrix with rank m, then the random vector generated by

$$\mathbf{Y} = \mathbf{AX}$$

has an m-dimensional Normal pdf with positive definite covariance matrix $\mathbf{Q}$ and mean vector $\boldsymbol{\beta}$ given respectively by

$$\begin{aligned} \mathbf{Q} &= \mathbf{AKA}^{\mathrm{T}} \\ \boldsymbol{\beta} &= \mathbf{A}\boldsymbol{\mu}. \end{aligned}$$

This problem concerns proving and applying this theorem.

(4a) Use the characteristic function of a n-dimensional Normal pdf

$$\Phi_{\mathbf{X}}(\boldsymbol{\omega}) = \Phi_{\mathbf{X}}(\omega_1, \ldots, \omega_n) = \exp\left\{ j\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{K}\boldsymbol{\omega} \right\},$$

to prove the theorem. (10 points)
Hint: Compute the pdf of $\mathbf{Y}$ from its characteristic function $\Phi_{\mathbf{Y}}(\tilde{\boldsymbol{\omega}}) = \Phi_{\mathbf{Y}}(\tilde{\omega}_1, \ldots, \tilde{\omega}_m)$.

Extra work-space 3

(4b) Consider the specific case of $n = m = 2$, and let

$$\mathbf{K} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Find a matrix $\mathbf{A}$ such that the random vector $\mathbf{Y} = \mathbf{AX}$ has *independent* components, each with zero mean and unit variance. (10 points)

Question N$\underline{\text{o}}$ 5: *Estimation of Nonrandom Parameters.* Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with common pdf

$$f_X(x) = \begin{cases} 0 & \text{if } x < a, \\ \frac{1}{b-a} & \text{if } a \le x \le b, \\ 0 & \text{if } b < x. \end{cases}$$

This problem is concerned with the estimation of the parameters $\mathbf{\Theta} = \begin{bmatrix} a \\ b \end{bmatrix}$ of the pdf $f_X(\cdot)$.

(5a) Show that the maximum likelihood estimator of $\mathbf{\Theta}$ is (4 points)

$$\widehat{\mathbf{\Theta}}_{\text{ML}}(X_1, \ldots, X_n) = \begin{bmatrix} X^{(1)} \\ X^{(n)} \end{bmatrix}, \qquad \text{where} \qquad \begin{array}{l} X^{(1)} = \min\{X_1, \ldots, X_n\}, \quad \text{and} \\ X^{(n)} = \max\{X_1, \ldots, X_n\}. \end{array}$$

Hint: This was assigned as a homework problem.

(5b) Compute $E\left[\widehat{\boldsymbol{\Theta}}_{\mathrm{ML}}\right]$ and show that $\widehat{\boldsymbol{\Theta}}_{\mathrm{ML}}$ is a biased estimator. (10 points)
Hint: You may wish to calculate the pdfs of $X^{(1)}$ and $X^{(n)}$.

Extra work-space 4

(5c) Construct an *unbiased* estimator of $\boldsymbol{\Theta}$ from $\begin{bmatrix} X^{(1)} \\ X^{(n)} \end{bmatrix}$ for $n > 1$. (6 points)

Extra work-space 5