ECE 520.651 Random Signal Analysis

Second (Final) Examination, Fall 2004
9:00 AM — 12:00 PM, December 17, 2004.

Name: ______________________________________

Read these instructions before starting the examination.

(i) This is an open-book examination. Use of the Stark and Woods textbook, Prof Papamarou’s notes, and homework solutions & other printed notes provided in class is permitted. Additional photocopied material from other books, your class notes or homework solutions, material obtained via the Internet etc. are not permitted.

(ii) Use of electronic calculators is permitted for numeric calculations only.

(iii) Show all your work clearly and concisely. Points may be deducted for illegible or unclear answers.

(iv) Write your answers in the space provided. Use the unprinted side of the pages for additional space.

(v) There are five mandatory questions for a total of 100 points and a bonus question for 20 points. Use the check-list below to keep track of your progress.

Best of luck!

Question No 1 (a) (b) /20 Points
Question No 2 (a) (b) (c) (d) /20 Points
Question No 3 (a) (b) (c) (d) /20 Points
Question No 4 (a) (b) /20 Points
Question No 5 (a) (b) (c) (d) (e) /20 Points
Question No 6 (a) (b) (c) Bonus /20 Points

TOTAL /100 Points
**Question No 1: Analysis of Random Sequences.** Consider an urn containing \( N \) balls, each marked with a unique integer from 1 to \( N \). Balls are drawn with replacement from this urn.

(1a) Due to repetitions in sampling with replacement, a sample of size \( x \) will in general contain less than \( x \) distinct integers. Let \( X[m] \) denote the sample size necessary for the acquisition of \( m \) distinct integers, \( 1 \leq m \leq N \). Note that \( X[1] = 1 \), since a distinct integer is always acquired at the first drawing, and that \( X[1] < X[2] < \ldots < X[N] \).

- Does the random sequence \( X[m] \) have independent increments? (3 points)
- Calculate the *mean* function \( \mu_X[m] \) of the random sequence \( X[m] \). (4 points)
- Show that for large \( N \), the expected sample size necessary for the acquisition of a fraction \( \alpha \) of the elements is approximately \( N \log \frac{1}{1-\alpha} \). (3 points)

Hint: \( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r} \approx \int_1^r \frac{1}{x}dx \).
(1b) Let $Y[n]$, $n = 1, 2, \ldots$, denote the highest integer observed in $n$ samples drawn with replacement as described above. Clearly, $1 \leq Y[1] \leq Y[2] \leq \ldots$, and the random sequence $Y[n]$ is bounded above by $N$.

- Compute the marginal pmf $P(Y[n] = k)$ of the random sequence. (4 points)
- Compute the mean function $\mu_{Y[n]}$ of the random sequence. (3 points)
- Show that for large $N$, the expected value of the highest integer observed in $n$ samples is approximately $\frac{n}{n+1}N$. (3 points)

Hint: $0 + 1 + 2^n + 3^n + \ldots + (N-1)^n \approx \int_0^N x^n \, dx$. 
**Question No 2: Markov Chains.** Let \( \{X[n], n = 0, 1, \ldots\} \) be a Markov chain taking values on a discrete set \( \mathcal{S} \), with

\[
\pi_0(i) = P(X[0] = i) \quad \text{and} \quad P_n(j|i) = P(X[n+1] = j|X[n] = i) \quad i, j \in \mathcal{S}.
\]

(2a) Show that for all \( n > 0 \) and \( x_0, \ldots, x_{n+1} \in \mathcal{S} \), (5 points)

\[
P(X[n+1] = x_{n+1}|X[n-1] = x_{n-1}, \ldots, X[0] = x_0) = P(X[n+1] = x_{n+1}|X[n-1] = x_{n-1})
\]

and compute this 2-step transition probability in terms of \( \pi_0 \) and the \( P_n \)’s.
(2b) Show that for all $0 \leq l < m < n$ and $x_l, x_m, x_n \in \mathcal{S}$, 

$$P(X[n] = x_n | X[m] = x_m, X[l] = x_l) = P(X[n] = x_n | X[m] = x_m).$$

**Remark:** This property of Markov chains is sometimes stated as “the future is independent of the past given the present.”
(2c) Is the random sequence $X[m], m = 1, \ldots, N,$ in Question (1a) a Markov chain? If it is, specify its state space $\mathcal{S}$, initial pmf $\pi_1(i)$, and transition probabilities $P_m(j|i)$, $i, j \in \mathcal{S}$. If it isn’t, show where the Markov property fails to hold. \hfill (5 points)

(2d) Is the random sequence $Y[n], n = 1, 2, \ldots$, in Question (1b) a Markov chain? If it is, specify its state space $\mathcal{S}$, initial pmf $\pi_1(i)$, and transition probabilities $P_n(j|i)$, $i, j \in \mathcal{S}$. If it isn’t, show where the Markov property fails to hold. \hfill (5 points)
Question No 3: Parameter Estimation. Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed random variables with common pdf

$$p_\theta(y) = \frac{e^{-\theta \theta^y}}{y!}, \quad y = 0, 1, \ldots,$$

parameterized by some $\Theta = \theta > 0$. Note that, by convention, $0! = 1$ and $0^0 = 1$.

(3a) Assuming $\theta$ is a fixed but unknown parameter, find its maximum likelihood estimate based on $Y_1, \ldots, Y_n$, and compute the bias and variance of your estimate. (6 points)
(3b) Compute, if applicable, the Cramér-Rao lower bound for the variance of unbiased estimates of $\theta$, or explain why it isn’t applicable. (4 points)
(3c) Assuming Θ is a random variable distributed uniformly on \((0,1]\), find the MAP estimate of Θ based on \(Y_1, \ldots, Y_n\). (5 points)
(3d) Assuming $\Theta$ is a random variable with pdf

$$w(\theta) = \begin{cases} \alpha e^{-\alpha \theta} & \theta > 0, \\ 0 & \text{otherwise}, \end{cases}$$

for a known $\alpha > 0$, find the MMSE estimate of $\Theta$ based on $Y_1, \ldots, Y_n$. (5 points)
**Question No 4:** Estimation of (Gaussian) Random Variables.

(4a) Let \( \mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n \) and \( \Theta \) be jointly Gaussian random vectors, such that \( \mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n \) are mutually independent and have zero means. Show that

\[
\hat{\theta}(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) = \sum_{k=1}^{n} E[\Theta | \mathbf{Z}_k] - (n - 1)E[\Theta]
\]

is the MMSE estimate of \( \Theta \) based on \( \mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n \).  

(10 points)
(4b) Consider the (scalar) linear dynamical system

\[
X_{k+1} = \frac{1}{2} X_k + U_k, \quad k = 0, 1, \ldots, \\
Y_k = X_k + V_k, \quad k = 0, 1, \ldots,
\]

where \( X_0 \) is a \( \mathcal{N}(0, \sigma^2) \) random variable, \( U_0, U_1, \ldots \) are i.i.d. \( \mathcal{N}(0, 1) \) random variables, \( V_0, V_1, \ldots \) are i.i.d. \( \mathcal{N}(0, 1) \) random variables, and all \( U_k \)'s, \( V_k \)'s and \( X_0 \) are independent of one another.

- Find the value of \( \sigma^2 \) such that the MMSE predictor of \( X_{t+1} \) based on \( Y_0, \ldots, Y_t \) is a time-invariant filter. \hspace{1cm} (5 points)

- With \( \sigma^2 \) chosen as above, write the recursion for \( \hat{X}_{t+1|t} \), i.e. provide a seed \( \hat{X}_{0|t-1} \) and, for \( t = 0, 1, \ldots \), write \( \hat{X}_{t+1|t} \) in terms of \( \hat{X}_{t|t-1} \) and \( Y_t \). \hspace{1cm} (5 points)
Question No 5: Binary Hypothesis Testing. Let $S$ and $N$ be independent $\mathbb{R}$-valued random variables with pdfs

$$p_S(s) = \begin{cases} 1 & s \in [0,1], \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad p_N(n) = \begin{cases} e^{-n} & n \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

(5a) Calculate the likelihood ratio for testing

$$H_1 : \ Y = S + N \quad \text{versus} \quad H_0 : \ Y = N.$$  

(4 points)

(5b) Sketch the likelihood ratio $L(y)$ versus $y \in \mathbb{R}$. Mark any salient points in the graph. If needed, evaluate $L(y)$ for specific values, e.g. $y = 0 \pm \epsilon$ and $y = 1 \pm \epsilon$.  

(4 points)
(5c) Characterize the acceptance region $\Gamma_1$ for the Bayes rule $\delta_{\pi_0}$ assuming equal priors and costs $C_{10} = \sqrt{c}$, $C_{01} = C_{00} = 1$ and $C_{11} = 0$. (4 points)

(5d) Compute the power of the level-$\alpha$ Neyman-Pearson test $\delta_{NP}$ for $\alpha \in (\frac{1}{2}, 1)$. (4 points)
(5e) Characterize the level-\(\alpha\) Neyman-Pearson test \(\delta_{NP}\) for \(\alpha \in (0, \frac{1}{e})\). (4 points)
Question No 6 Bonus Question on Various Topics.

(6a) Channel Equalization. Let $X(t)$ and $N(t)$, $t \in \mathbb{R}$, be orthogonal, zero-mean, jointly wide-sense stationary random processes. $X(t)$ represents the input signal to a linear time-invariant (LTI) channel with impulse response $h(t)$, and $N(t)$ an additive noise:

$$Y(t) = h(t) \ast X(t) + N(t).$$

We wish to design a LTI filter with impulse response $g(t)$ to process $Y(t)$, so that

$$\tilde{X}(t) = g(t) \ast Y(t)$$

statistically minimizes

$$\Delta(t) = \tilde{X}(t) - X(t),$$

the error in the reconstruction of the signal $X(t)$. For instance, we may wish to minimize $E[|\Delta^2(t)|] = R_{\Delta \Delta}(0) = \frac{1}{2\pi} \int S_{\Delta \Delta}(\omega)d\omega$. To this end, we wish to compute the power spectral density of the reconstruction error.

- Write $S_{YY}(\omega)$ in terms of $S_{XX}(\omega)$, $S_{NN}(\omega)$ and $H(\omega)$. (3 bonus points)
- Write $S_{\tilde{X}X}(\omega)$ in terms of $S_{XX}(\omega)$, $S_{NN}(\omega)$, $H(\omega)$ and $G(\omega)$. (4 bonus points)
- Write $S_{\Delta \Delta}(\omega)$ in terms of $S_{XX}(\omega)$, $S_{NN}(\omega)$, $H(\omega)$ and $G(\omega)$. (4 bonus points)
(6b) *H.F.7 from Poor Revisited.* Consider the hypothesis pair

\[ H_0 : Y_k = N_k, \quad k = 1, \ldots, n, \quad \text{versus} \quad H_1 : Y_k = S + N_k, \quad k = 1, \ldots, n, \]

where \( N_1, \ldots, N_n \) and \( S \) are mutually independent random variables with common pdf

\[
p(x) = \begin{cases} 
  e^{-x} & \text{if } x \geq 0, \\
  0 & \text{otherwise.}
\end{cases}
\]

Check if \( T(Y_1^n) = \min\{Y_1, \ldots, Y_n\} \) is a sufficient statistic for testing \( H_1 \) versus \( H_0 \). Does your answer depend on the type of test being constructed? (5 bonus points)
(6c) *Polya’s Contagion Models Revisited.* An urn contains $b$ black and $r$ red balls. A ball is drawn at random. It is replaced and, moreover, $c$ balls of the color drawn are added to the urn. A new random drawing is made from the urn, now containing $b + r + c$ balls, and this procedure is repeated.

Let $X_m \in \{\text{black, red}\}$, $m = 1, 2, \ldots$, denote the color of the ball at the $m$-th drawing. The following is offered as proof by induction that $P(X_m = \text{black}) = \frac{b}{b+r}$ for all $m$.

Assume that $P(X_n = \text{black}) = \frac{b}{b+r}$ for some $n \geq 1$. Furthermore, let $b_n$ be the number of black balls in the urn before $X_n$ is drawn. Then

$$P(X_n = \text{black}) = \frac{b}{b+r} \Rightarrow \frac{b_n}{b+r + (n-1)c} = \frac{b}{b+r} \Rightarrow b_n(b+r) = b(b+r + (n-1)c).$$

Next,

$$P(X_{n+1} = \text{black}) = P(X_{n+1} = \text{black} | X_n = \text{black})P(X_n = \text{black}) + P(X_{n+1} = \text{black} | X_n = \text{red})P(X_n = \text{red})$$

$$= \frac{b_n + c}{b+r + nc} \times \frac{b}{b+r} + \frac{b_n}{b+r + nc} \times \frac{r}{b+r}$$

$$= \frac{bb_n + bc + b_nr}{(b+r + nc)(b+r)} = \frac{b_n(b+r) + bc}{(b+r + nc)(b+r)}$$

$$= \frac{b(b+r + (n-1)c) + bc}{(b+r + nc)(b+r)} = \frac{b}{b+r}.$$  

Since it is easy to show that $P(X_1 = \text{black}) = \frac{b}{b+r}$, and

$$P(X_n = \text{black}) = \frac{b}{b+r} \quad \Rightarrow \quad P(X_{n+1} = \text{black}) = \frac{b}{b+r}, \quad n \geq 1,$$

it follows by induction that $P(X_m = \text{black}) = \frac{b}{b+r}$ for all $m$.

What, if anything, is wrong with this proof? (4 bonus points)