ECE 520.651
Random Signals Analysis

Midterm Examination #1

9:00 — 10:45 AM, October 30, 2003.

Name: ____

Read these instructions before starting the examination.

(i) This is a open-book examination. Use of the Stark and Woods textbook is permitted. Photocopied material from other books, handwritten/class notes, homework soultions, *etc.* are not permitted.

(ii) Use of electronic calculators is permitted for numeric calculations only.

(iii) Show all your work clearly and concisely. Points may be deducted for illegible or unclear answers.

(iv) Provide answers in the space provided. Use the unprinted side of the pages for additional space.

(v) There are five mandatory questions for a total of 100 points.

Best of luck!

Question № 1	/20 Points
Question № 2	/20 Points
Question № 3	/20 Points
Question № 4	/20 Points
Question № 5	/20 Points
TOTAL	/100 Points

Question No̲ 1: A dart is flung at a circular target of radius $r = 9$. Consider the hitting point as the outcome of a random experiment and assume that the dart is guaranteed to hit the target somewhere. Assume further that a player's score depends on where the dart hits the target.

(1a) Define a suitable sample space $\{\Omega, \mathcal{F}\}$ for modeling the outcome ξ of the random experiment.

(1b) Assuming the dart is equally likely to land on any point on the target, define a probability measure on your chosen $\{\Omega, \mathcal{F}\}$. In particular, explain how the probability $P(A)$ of a set $A \in \mathcal{F}$ will be obtained under this probability measure. You may invent reasonable notation to denote the area of A, the circumference of A, the number of points in A, etc as needed to compute $P(A)$.

(1c) Let $X(\xi)$ denote the distance of the hit from the center of the target. Compute and sketch the cumulative distribution function $F_X(x)$ and, if it exists, the probability density function $f_X(x)$ of the random variable X.

Question No̲ 2

(2a) Consider a sequence of independent and identically distributed $\mathbb{R}$-valued random variables $X_1, X_2, \ldots$, with mean μ, and let N be another random variable independent of the X_i and taking values on non-negative integers. Let

$$S = X_1 + X_2 + \ldots + X_N.$$

Show that $E[S|N] = \mu N$ and hence deduce that $E[S] = \mu E[N]$.

(2b) Let X be a $\mathbb{R}$-valued random variable with probability density function

$$f_X(x) = \lambda e^{-\lambda x} u(x).$$

Compute the probability density function of the *residual* value of X given that X exceeds some value α, i.e. define a random variable $Y = X - \alpha$, and compute the *conditional* probability density function $f_Y(y|X \geq \alpha)$.

Question No 3: Consider the real-valued random variables X and Y with joint pdf

$$f_{XY}(x, y) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(x^2+y^2\right)}.$$

Let Z and W be defined as

$$\begin{aligned} Z &= X\cos\theta + Y\sin\theta \\ W &= X\sin\theta - Y\cos\theta, \end{aligned} \tag{1}$$

for some known constant θ.

(3a) Compute the joint pdf of Z and W.

(3b) Are X and Y independent?

(3c) Are Z and W indepednent?

Question No̲ 4: Let $W[n]$ be an i.i.d. random sequence with mean 0 and variance σ_W^2 defined for $-\infty < n < +\infty$. For an appropriate choice of ρ, let the stationary random sequence $X[n]$ satisfy the causal linear constant-coefficient difference equation

$$X[n] = \rho X[n-1] + W[n], \qquad -\infty < n < +\infty.$$

(4a) Show that $X[n-1]$ and $W[n]$ are independent for every n.

(4b) Show that the characteristic equation $\Phi_X(\omega)$ of the marginal pdf of X is related to the characteristic equation of W as

$$\Phi_X(\omega) = \Phi_X(\rho\omega)\Phi_W(\omega).$$

(4c) Assume that W is Gaussian and compute $\Phi_X(\omega)$.

(4d) Determine the variance σ_X^2 of X.

Question No̲ 5: A Cauchy random variable has a pdf

$$f_X(x) = \frac{1}{\pi\left(1+x^2\right)}, \qquad -\infty < x < +\infty.$$

Let $X_1, X_2, \ldots,$ be i.i.d. random variables with the Cauchy distribution and let

$$Y[n] = \frac{\pi}{n} X^{(n)} = \frac{\pi}{n} \times \max\{X_1, X_2, \ldots, X_n\}.$$

(5a) Show that the random sequence $Y[n]$ converges in distribution, the limiting distribution function being given by $H(y) = u(y)e^{-\frac{1}{y}}$.

(5b) Does $Y[n]$ converge in any other sense?